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# THE CONTROL TRANSMUTATION METHOD AND THE COST OF FAST CONTROLS\*

LUC MILLER†

**Abstract.** In this paper, the null controllability in any positive time  $T$  of the first-order equation (1)  $\dot{x}(t) = e^{i\theta}Ax(t) + Bu(t)$  ( $|\theta| < \pi/2$  fixed) is deduced from the null controllability in some positive time  $L$  of the second-order equation (2)  $\ddot{z}(t) = Az(t) + Bv(t)$ . The differential equations (1) and (2) are set in a Banach space,  $B$  is an admissible unbounded control operator, and  $A$  is a generator of cosine operator function.

The control transmutation method explicits the input function  $u$  of (1) in terms of the input function  $v$  of (2):  $u(t) = \int_{\mathbb{R}} k(t, s)v(s) ds$ , where the compactly supported kernel  $k$  depends on  $T$  and  $L$  only. It proves roughly that the norm of a  $u$  steering the system (1) from an initial state  $x(0) = x_0$  to the final state  $x(T) = 0$  grows at most like  $\|x_0\| \exp(\alpha_* L^2/T)$  as the control time  $T$  tends to zero. (The rate  $\alpha_*$  is characterized independently by a one-dimensional controllability problem.)

In the applications to the cost of fast controls for the heat equation,  $L$  is roughly the length of the longest ray of geometric optics which does not intersect the control region.

**Key words.** Controllability, fast controls, control cost, transmutation, cosine operator function, heat equation, linear Ginzburg-Landau equation.

**AMS subject classifications.** 93B05, 93B17, 47D09

**1. Introduction.** This paper concerns the relationship between the null-controllability of the following first and second order controllable systems:

$$\dot{x}(t) = e^{i\theta}Ax(t) + Bu(t) \quad (t \in \mathbb{R}_+), \quad x(0) = x_0, \quad (1.1)$$

$$\ddot{z}(t) = Az(t) + Bv(t) \quad (t \in \mathbb{R}), \quad z(0) = z_0, \quad \dot{z}(0) = 0, \quad (1.2)$$

where  $x$  and  $z$  are the systems trajectories in the Banach space  $X$ ,  $x_0$  and  $z_0$  are initial states,  $u$  and  $v$  are input functions with values in the Banach space  $U$ ,  $A$  is an unbounded generator,  $B$  is an unbounded control operator,  $\theta$  is a given angle in  $]-\pi/2, \pi/2[$ , each dot denotes a derivative with respect to the time  $t$  and  $\mathbb{R}_+ = [0, \infty)$  (the detailed setting is given in §2).

Equation (1.1) with  $u = 0$  describes an irreversible system (always smoothing) and we think of it as a parabolic distributed system with infinite propagation speed. Equation (1.2) with  $v = 0$  describes a reversible system (e.g. conservative) and we think of it as a hyperbolic distributed system with finite propagation speed.

For example, if  $A$  is the negative Laplacian on a Euclidean region and the input function is a locally distributed boundary value set by  $B$ , then (1.2) is a boundary controlled scalar wave equation and (1.1) with  $\theta = 0$  is a boundary controlled heat equation (§6 elaborates on this example). Thus, this paper applies to the controllability of the diffusion equation that models the propagation of heat in a media at rest when the temperature does not vary too much. This diffusion equation is also a

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\* In the final version of this preprint, accepted for publication in SIAM J. Control and Optimization on January 26, 2006, remark 6.5 and the reference to [16] in footnote 3 have been erased. Since the previous version, posted on [arxiv.org](http://arxiv.org) and submitted to SIAM on February 4, 2004, remarks 6.3, 6.4, 6.5 and Lemma 2.4 have been added. The third paragraph and the footnotes added to the introduction are only mildly relevant.

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simplified model for various physical phenomena in neutronics, viscous fluids mechanics and electromagnetism when  $A$  is a more general coercive symmetric second-order elliptic operator (cf. §I.B.1.1 in [11]). If  $\theta \neq 0$ , then (1.1) interpolates between the heat equation and the Schrödinger equation. Thus, this paper applies to the controllability of the linear part of the complex Ginzburg-Landau equation which describes a vast variety of physical phenomena from nonlinear waves to second-order phase transitions, from superconductivity, superfluidity, and Bose-Einstein condensation to liquid crystals and strings in field theory (cf. [2]). In particular, the one-dimensional linearized Ginzburg-Landau model considered in [1] enters this setting. If  $\pi/2 - \theta$  is a small positive constant, then (1.1) is a viscous perturbation of the Schrödinger equation which appears naturally in the building of stable numerical schemes (cf. [19]). Although conservative physical systems (e.g. the linear system of elasticity, Maxwell's equations, cf. §I.B.1.2 in [11]) provide examples of (1.2) for which some controllability results are known, we have not found one yet for which the corresponding first-order system (1.1) is relevant to physics or engineering (besides the scalar wave equation considered in §6).

This paper presents the *control transmutation method* (cf. [18] for a survey on transmutations in other contexts) which can be seen as a shortcut to Russell's famous harmonic analysis method in [27]. It consists in explicitly constructing controls  $u$  in any time  $T$  for the heat-like equation (1.1) in terms of controls  $v$  in time  $L$  for the corresponding wave-like equation (1.2), i.e.  $u(t) = \int_{\mathbb{R}} k(t, s)v(s) ds$ , where the compactly supported kernel  $k$  depends on  $T$  and  $L$ <sup>1</sup>. It proves that the exact controllability of (1.2) in some time  $L$  implies the null controllability of (1.1) in any time with a relevant upper bound on the cost of controls for (1.1) as  $T$  tends to 0, in short *the cost of fast controls* (as in the title of this paper and [31]). Thanks to the geodesic condition of Bardos-Lebeau-Rauch (cf. [6]) for the controllability of the wave equation, the application of this method to the boundary controllability of the heat equation (cf. §6) yields new geometric bounds on the cost of fast controls, extending the results of [23] on internal controllability<sup>2</sup>. This method applies as soon as the wave equation is controllable, no matter how singular the coefficients in  $A$  are (e.g. discontinuous coefficients in transmission problems in [12], coefficients of bounded variation in [16]) but it does not seem to adapt to time-dependent coefficients. The companion paper [22] concerns the quite different case  $|\theta| = \pi/2$  (in particular there is no smoothing effect) and applies to the Schrödinger equation, which is the control problem to which a transmutation method was first applied by Phung in [26].

The relationship between first and second order controllable systems has been investigated in previous papers, always with  $\theta = 0$  and the additional initial data  $\dot{z}(0) = z_1$  in  $X$  (i.e. considering trajectories of (1.2) in the state space  $X \times X$ ). In [13], it is proved that the approximate controllability of (1.2) with  $\dot{z}(0) = z_1$  in some time implies the approximate controllability of (1.1) for any time (the control transmutation method yields an alternative proof), and proves the converse under

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<sup>1</sup>N.b. although  $L$  denotes a time here, it also denotes a length in the construction of  $k$  and in the main application to the wave and heat equations; the notation is meant as a reminder of this key fact of the method.

<sup>2</sup>N.b. the controllability of the heat equation holds even if the geometry prevents the controllability of the wave equation but in that case no explicit geometric bound on the cost of fast controls is known. For interior controllability, a bound of the form  $\exp(C_\beta/T^\beta)$  for any  $\beta > 1$  is proved in [25] but the positive constant  $C_\beta$  is not explicit. A better bound of the form  $\exp(C_1/T)$  is proved in [15], at least in the Euclidean case (cf. remark 6.4), but the positive constant  $C_1$  is not explicit either.

some assumptions on the spectrum of  $A$ . The converse is investigated further in [33] (in Hilbert spaces) and [32]. In a restricted setting, the null controllability of (1.1) was deduced from the exact controllability of (1.2) with  $\dot{z}(0) = z_1$  in [27] and [28] by the indirect method of bi-orthogonal bases<sup>3</sup>.

The study of the cost of fast controls was initiated by Seidman in [29] with a result on the heat equation obtained by Russell's method. Seidman also obtained results on the Schrödinger equation by working directly on the corresponding window problem for series of complex exponentials (see [24] for improvements and references). With collaborators, he later treated the case of finite dimensional linear systems (cf. [31]) and generalized the window problem to a larger class of complex exponentials (cf. [30]). Fernández-Cara and Zuazua obtained results on the cost of fast interior controls for the heat equation discussed in remark 6.4. Since their approach is based on global Carleman estimates (cf. [17]) it should extend to boundary controllability, and to a very large class of equations with variable coefficients possibly depending on time as well. The control transmutation method generalizes upper bounds on the cost of fast controls from the one-dimensional setting (which reduces to a window problem) to the general setting which we specify in the next section. Some open problems are stated in remarks 3.3 and 6.4.

**2. The setting.** We assume that  $A$  is the generator of a strongly continuous cosine operator function  $\text{Cos}$  (i.e. the second-order Cauchy problem for  $\ddot{z}(t) = A(t)z$  is well posed and  $\text{Cos}$  is its propagator). For a textbook presentation of cosine operator functions, we refer to chap. 2 of [14] or §3.14 of [3]. The associated sine operator function is  $\text{Sin}(t) = \int_0^t \text{Cos}(s)ds$  (with the usual Bochner integral).  $\text{Cos}$  and  $\text{Sin}$  are strongly continuous functions on  $\mathbb{R}$  of bounded operators on  $X$ . Moreover  $A$  generates a holomorphic semigroup  $T$  of angle  $\pi/2$  (cf. th. 3.14.17 of [3]). In particular  $S(t) = T(e^{i\theta}t)$  defines a strongly continuous semigroup  $(S(t))_{t \in \mathbb{R}_+}$  of bounded operators on  $X$ . In this setting, for any source term  $f \in L^1_{\text{loc}}(\mathbb{R}, X)$ , for any initial data  $x_0, z_0$  and  $z_1$  in  $X$ , the inhomogeneous first and second order Cauchy problems

$$\dot{x}(t) = e^{i\theta}Ax(t) + f(t) \quad (t \in \mathbb{R}_+), \quad x(0) = z_0, \quad (2.1)$$

$$\ddot{z}(t) = Az(t) + f(t) \quad (t \in \mathbb{R}), \quad z(0) = z_0, \quad \dot{z}(0) = z_1, \quad (2.2)$$

have unique mild solutions  $x \in C^0(\mathbb{R}_+, X)$  and  $z \in C^0(\mathbb{R}, X)$  defined by:

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s)dt, \quad z(t) = \text{Cos}(t)z_0 + \text{Sin}(t)z_1 + \int_0^t \text{Sin}(t-s)f(s)dt.$$

*Remark 2.1.* When  $A$  is a negative self-adjoint unbounded operator on a Hilbert space,  $T$ ,  $\text{Cos}$  and  $\text{Sin}$  are simply defined by the functional calculus as  $T(t) = \exp(tA)$ ,  $\text{Cos}(t) = \cos(t\sqrt{-A})$  and  $\text{Sin}(t) = (\sqrt{-A})^{-1} \sin(t\sqrt{-A})$ .

Following [34], we now make natural assumptions on  $B$  for any initial data in the state space  $X$  to define a unique continuous trajectory of each system (1.1) and (1.2). Let  $X_{-1}$  be the completion of  $X$  with respect to the norm  $\|x\|_{-1} = \|(A - \beta)^{-1}x\|$

<sup>3</sup> N.b. the systematic approach to controllability based on bi-orthogonal bases (cf. [4]) does not apply in our general setting where  $A$  may not have eigenfunctions (e.g. on some unbounded domains). When it does apply, it builds on some knowledge about the eigenvalues and the analysis of some exponential series, whereas the control transmutation method *per se* does not (cf. remark 3.3). E.g. in the abstract presentation of Russell's method in §2 of [16], the eigenvalues are assumed to grow quadratically. A more detailed comparison can be found in [23], in the specific case which lead the author to this new method.

for some  $\beta \in \mathbb{C}$  outside the the spectrum of  $A$ .  $X_{-1}$  is also the dual of the space  $X_1$  defined as the domain  $D(A)$  with the norm  $\|x\|_1 = \|(A - \beta)x\|$ . We assume that  $B \in \mathcal{L}(U, X_{-1})$  is an *admissible unbounded control operator* in the following sense:

$$\forall t > 0, \forall u \in L^2([0, t]; U), \int_0^t S(s)Bu(s)ds \in X \text{ and } \int_0^t \text{Sin}(s)Bu(s)ds \in X. \quad (2.3)$$

In this setting, for any  $x_0$  and  $z_0$  in  $X$ , for any  $u$  and  $v$  in  $L^2_{\text{loc}}(\mathbb{R}_+; U)$ , the unique solutions  $x$  and  $z$  in  $C^0(\mathbb{R}; X)$  of (1.1) and (1.2) respectively are defined by:

$$x(t) = S(t)x_0 + \int_0^t S(t-s)Bu(s)dt, \quad z(t) = \text{Cos}(t)z_0 + \int_0^t \text{Sin}(t-s)Bv(s)ds. \quad (2.4)$$

The natural notions of controllability cost for the linear systems (1.1) and (1.2) are:

**DEFINITION 2.2.** *The system (1.1) is null-controllable in time  $T$  if for all  $x_0$  in  $X$ , there is a  $u$  in  $L^2(\mathbb{R}_+; U)$  such that  $u(t) = 0$  for  $t > T$  and  $x(T) = 0$ . The controllability cost for (1.1) in time  $T$  is the smallest positive constant  $\kappa_{1,T}$  such that, for all  $\phi_0$ , there is such a  $u$  satisfying:  $\int_0^T \|u(t)\|^2 dt \leq \kappa_{1,T} \|x_0\|^2$ .*

*The system (1.2) is null-controllable in time  $T$  if for all  $z_0$  in  $X$ , there is a  $v$  in  $L^2(\mathbb{R}_+; U)$  such that  $v(t) = 0$  for  $t > T$  and  $z(T) = \dot{z}(T) = 0$ . The controllability cost for (1.2) in time  $T$  is the smallest positive constant  $\kappa_{2,T}$  such that, for all  $z_0$ , there is such a  $v$  satisfying:  $\int_0^T \|v(t)\|^2 dt \leq \kappa_{2,T} \|z_0\|^2$ .*

**Remark 2.3.** *Equivalently, for all  $x_T$  in  $S(T)X$ , there is a  $u$  in  $L^2(0, T; U)$  such that  $x(0) = 0$  and  $x(T) = x_T$ , and, for all  $z_0, z_1, z_2$  and  $z_3$  in  $X$ , there is a  $v$  in  $L^2(0, T; U)$  such that  $(z(0), \dot{z}(0)) = (z_0, z_1)$  and  $(z(T), \dot{z}(T)) = (z_2, z_3)$ . This is the usual notion of exact controllability for (1.2) (cf. lemma 2.4).*

Although the notion of null-controllability for (1.2) in def. 2.2 seems weaker than usual, the following lemma proves that it is equivalent to the usual notion.

**LEMMA 2.4.** *If (1.2) is null-controllable in time  $T$  in the sense of def. 2.2 then, for all  $z_0$  and  $z_1$  in  $X$ , there is an input  $v$  in  $L^2(\mathbb{R}_+; U)$  such that  $v(t) = 0$  for  $t > T$  and the solution of  $\ddot{z}(t) = Az(t) + Bv(t)$  ( $t \in \mathbb{R}$ ) with initial condition  $(z(0), \dot{z}(0)) = (z_0, z_1)$  satisfies the final condition  $(z(T), \dot{z}(T)) = (0, 0)$ .*

*Proof.* Assume that the system (1.2) satisfies def. 2.2 and let  $z_0$  and  $z_1$  be in  $X$ . We shall build an input  $w$  in  $L^2(0, T; U)$  such that the solution  $y$  of  $\ddot{y}(t) = Ay(t) + Bw(t)$  with  $(y(0), \dot{y}(0)) = (0, z_1)$  satisfies  $(y(T), \dot{y}(T)) = (y_0, 0)$  for some  $y_0 \in X$ . Let  $v_1$  be the input  $v$  in def. 2.2 and let  $v_2$  be the input  $v$  in def. 2.2 with  $z_0$  replaced by  $y_0$ . Due to the linearity of the system, the input  $v_1(t) - v_2(T-t) + w(t)$  steers the system from  $(z_0, z_1)$  to  $(0, 0)$  as required to complete the proof of the lemma. The rest of the proof builds  $w$  and  $y$ .

Applying def. 2.2 with  $z_0$  replaced by  $z_1$ , yields an input  $v$  and the solution:

$$z(s) = \text{Cos}(s)z_1 + \int_0^s \text{Sin}(r)Bv(s-r)dr, \quad z(T) = \dot{z}(T) = 0.$$

We define  $w$  and  $y$  by  $w(t) = \int_0^t v(s)ds$  and  $y(t) = \int_0^t z(s)ds$  for  $t \in [0, T]$ . The final condition  $\dot{y}(T) = z(T) = 0$  is immediately satisfied. Since  $\text{Sin}(t) = \int_0^t \text{Cos}(s)ds$ :

$$\begin{aligned} y(t) - \text{Sin}(t)z_1 &= \int_0^t \int_0^s \text{Sin}(r)Bv(s-r)dr ds = \int_0^t \int_r^t \text{Sin}(r)Bv(s-r)ds dr \\ &= \int_0^t \text{Sin}(r)B \int_r^t v(s-r)ds dr = \int_0^t \text{Sin}(r)Bw(t-r)dr. \end{aligned}$$

The exchange of integrals in the second step is justified since  $(s, r) \mapsto \sin(r)Bv(s-r)$  is continuous with values in  $X_{-1}$  (Cos and Sin extend by density to strongly continuous functions on  $\mathbb{R}$  of bounded operators on  $X_{-1}$  since they commute with  $A - \beta$ ). This integral formula means that  $y$  is the mild solution of  $\ddot{y}(t) = Ay(t) + Bw(t)$  with initial condition  $(y(0), \dot{y}(0)) = (0, z_1)$ .  $\square$

**3. The results and the method.** Our estimate of the cost of fast controls for (1.1) builds, through the control transmutation method, on the same estimate for a simple system of type (1.1), i.e. on a segment  $[0, L]$  with Dirichlet ( $N = 0$ ) or Neumann ( $N = 1$ ) condition at the left end controlled at the right end through a Dirichlet condition:

$$\partial_t \phi = e^{i\theta} \partial_s^2 \phi \text{ on } ]0, T[ \times ]0, L[, \quad \partial_s^N \phi|_{s=0} = 0, \quad \phi|_{s=L} = u, \quad \phi|_{t=0} = \phi_0. \quad (3.1)$$

With the notations of §2,  $x = \phi$ ,  $A = \partial_s^2$  on  $X = L^2(0, L)$  with  $D(A) = \{f \in H^2(0, L) \mid \partial_s^N f(0) = f(L) = 0\}$ ,  $\|\cdot\|_1$  with  $\beta = 0$  is the homogeneous Sobolev  $\dot{H}^2(0, L)$  norm, and  $B$  on  $U = \mathbb{C}$  is the dual of  $C \in \mathcal{L}(X_1; U)$  defined by  $Cf = \partial_s f(L)$ .

It is well-known that the controllability of this system reduces by spectral analysis to classical results on nonharmonic Fourier series. The following upper bound for the cost of fast controls, proved in §4, is an application of a refined result of Avdonin-Ivanov-Seidman in [30].

**THEOREM 3.1.** *There are positive constants  $\alpha$  and  $\gamma$  such that, for all  $N \in \{0, 1\}$ ,  $L > 0$ ,  $T \in ]0, \inf(\pi, L)^2]$ , the controllability cost  $\kappa_{L,T}$  of the system (3.1) satisfies:  $\kappa_{L,T} \leq \gamma \exp(\alpha L^2/T)$ .*

This theorem leads to a definition of the optimal fast control cost rate for (3.1):

**DEFINITION 3.2.** *The rate  $\alpha_*$  is the smallest positive constant such that for all  $\alpha > \alpha_*$  there exists  $\gamma > 0$  satisfying the property stated in theorem 3.1.*

**Remark 3.3.** *Computing  $\alpha_*$  is an interesting open problem and its solution does not have to rely on the analysis of series of complex exponentials. The best estimate so far is  $\alpha_* \in [1/2, 4(36/37)^2]$  for  $\theta = 0$  (cf. [23]).*

Our main result is a generalization of theorem 3.1 to the first-order system (1.1) under some condition on the second-order system (1.2):

**THEOREM 3.4.** *If the system (1.2) is null-controllable for times greater than  $L_*$ , then the system (1.1) is null-controllable in any time  $T$ . Moreover, the controllability cost  $\kappa_{1,T}$  of (1.1) satisfies the following upper bound (with  $\alpha_*$  defined above):*

$$\limsup_{T \rightarrow 0} T \ln \kappa_{1,T} \leq \alpha_* L_*^2. \quad (3.2)$$

**Remark 3.5.** *The upper bound (3.2) means that the norm of an input function  $u$  steering the system (1.1) from an initial state  $x_0$  to zero grows at most like  $\gamma \|x_0\| \exp(\alpha L^2/(2T))$  as the control time  $T$  tends to zero (for any  $\alpha_*$  and some  $\gamma > 0$ ). The falsity of the converse of the first statement in th. 3.4 is well-known, e.g. in the more specific setting of §6.*

**Remark 3.6.** *As observed in [9], (3.2) yields a logarithmic modulus of continuity for the minimal time function  $T_{\min} : X \rightarrow [0, +\infty)$  of (1.1); i.e.  $T_{\min}(x_0)$ , defined as the infimum of the times  $T > 0$  for which there is a  $u$  in  $L^2(\mathbb{R}; Y)$  such that  $\int_0^T \|u(t)\|^2 dt \leq 1$ ,  $u(t) = 0$  for  $t \notin [0, T]$  and  $x(T) = 0$ , satisfies: for all  $\alpha > \alpha_*$ , there is a  $c > 0$  such that, for all  $x_0$  and  $x'_0$  in  $X$  with  $\|x_0 - x'_0\|$  small enough,  $|T_{\min}(x_0) - T_{\min}(x'_0)| \leq \alpha L_*^2 / \ln(c/\|x_0 - x'_0\|)$ .*

It is well known that the semigroup  $T$  can be expressed as an integral over the cosine operator function  $\text{Cos}$  (cf. the second proof of th. 3.14.17 in [3]):

$$\forall x \in X, \forall t \in \mathbb{C} \text{ s.t. } |\arg t| < \pi/2, \quad T(t)x = \int k(t, s) \text{Cos}(s)x ds, \quad (3.3)$$

where  $k(0, s) = \delta(s)$  and  $k(t, s) = \exp(-s^2/(4t))/\sqrt{\pi t}$  for  $\text{Re } t > 0$ . This equation has been referred to as the abstract Poisson or Weierstrass formula. Starting with the observation that  $k$  is the fundamental solution of the heat equation on the line, i.e.  $k$  is the solution of  $\partial_t k = \partial_s^2 k$  with the Dirac measure at the origin as initial condition, the *transmutation control method* consists in replacing the kernel  $k$  in (3.3) by some *fundamental controlled solution* on the segment  $[-L, L]$  controlled at both ends (cf. (5.8)). The one dimensional th. 3.1 is used to construct this fundamental controlled solution in §4 and the transmutation is performed in §5.

**4. The fundamental controlled solution.** This section begins with an outline of the standard application of [30] to the proof of th. 3.1. Following closely §5 of [23], the rest of the section outlines the construction of a “fundamental controlled solution”  $k$  in the following sense, where  $\mathcal{D}'(\mathcal{O})$  denotes the space of distributions on the open set  $\mathcal{O}$  endowed with the weak topology,  $\mathcal{M}(\mathcal{O})$  denotes the subspace of Radon measures on  $\mathcal{O}$ , and  $\delta$  denotes the Dirac measure at the origin:

DEFINITION 4.1. *The distribution  $k \in C^0([0, T]; \mathcal{M}([-L, L]))$  is a fundamental controlled solution for (4.1) at cost  $(\gamma, \alpha)$  if*

$$\partial_t k = e^{-i\theta} \partial_s^2 k \quad \text{in } \mathcal{D}'([0, T[ \times ]-L, L[) , \quad (4.1)$$

$$k|_{t=0} = \delta \quad \text{and} \quad k|_{t=T} = 0 , \quad (4.2)$$

$$\|k\|_{L^2([0, T[ \times ]-L, L[)}^2 \leq \gamma e^{\alpha L^2/T} . \quad (4.3)$$

The operator  $A$  defined at the beginning of §3 is negative self-adjoint on the Hilbert space  $L^2(0, L)$ . It has a sequence  $\{\mu_n\}_{n \in \mathbb{N}^*}$  of negative decreasing eigenvalues and an orthonormal Hilbert basis  $\{e_n\}_{n \in \mathbb{N}^*}$  in  $L^2(0, L)$  of corresponding eigenfunctions. Explicitly:  $\sqrt{-\mu_n} = (n + \nu)\pi/L$  with  $\nu = 0$  for  $N = 0$  (Dirichlet) and  $\nu = 1/2$  for  $N = 1$  (Neumann). First note that th. 3.1 can be reduced to the case  $L = \pi$  by the rescaling  $(t, s) \mapsto (\sigma^2 t, \sigma s)$  with  $\sigma = L/\pi$ . In terms of the coordinates  $c = (c_n)_{n \in \mathbb{N}^*}$  of  $A^{N/2} f_0$  in the Hilbert basis  $(e_n)_{n \in \mathbb{N}^*}$  where  $f_0$  is the initial state of the dual observability problem, th. 3.1 with  $L = \pi$  reduces by duality to the following window problem:  $\exists \alpha > 0, \exists \gamma > 0, \forall T \in ]0, \pi^2]$ ,

$$\forall c \in l^2(\mathbb{N}^*), \quad \sum_{n \in \mathbb{N}^*} |c_n|^2 \leq \gamma e^{\alpha \pi^2/T} \int_0^T |F(t)|^2 dt \quad \text{where } F(t) = \sum_{n=1}^{\infty} c_n e^{\exp(i\theta)\mu_n t} .$$

Since this results from th. 1 of [30] with  $\lambda_n \sim i e^{i\theta} n^2$  as in §5:2 of [30], the proof of th. 3.1 is completed.

Now we consider a system governed by the same equation as (3.1) but on the twofold segment  $[-L, L]$  controlled at both ends:

$$\partial_t \phi - e^{-i\theta} \partial_s^2 \phi = 0 \quad \text{in } ]0, T[ \times ]-L, L[, \quad \phi|_{s=\pm L} = u_{\pm}, \quad \phi|_{t=0} = \phi_0, \quad (4.4)$$

with initial state  $\phi_0 \in L^2(0, L)$ , input functions  $u_-$  and  $u_+$  in  $L^2(0, T)$ . As in proposition 5.1 of [23], applying th. 3.1 with  $N = 0$  to the odd part of  $\phi_0$  and with  $N = 1$  to

the even part of  $\phi_0$  proves that the controllability cost of (4.4) is not greater than the controllability cost of (3.1) and therefore satisfies the same estimate stated in th. 3.1. As in proposition 5.2 of [23], we may now combine successively the smoothing effect of (4.4) with no input (i.e.  $u_+ = u_- = 0$ ) and this controllability cost estimate (plugged into the integral formula expressing  $\phi$  in terms of  $\phi_0$  and  $u_{\pm} = \phi_{\gamma s = \pm L}$ ) to obtain:

PROPOSITION 4.2. *For all  $\alpha > \alpha_*$ , there exists  $\gamma > 0$  such that for all  $L > 0$  and  $T \in ]0, \inf(\pi/2, L)^2]$  there is a fundamental controlled solution for (4.4) at cost  $(\gamma, \alpha)$  (cf. def. 4.1).*

### 5. The transmutation of second-order controls into first-order controls.

In this section we prove th. 3.4.

Let  $x_0 \in X$  be an initial state for (1.1) and let  $L > L_*$ . Let  $z \in C^0(\mathbb{R}_+; X)$  and  $v \in L^2(\mathbb{R}_+; U)$  be the solution and input function obtained by applying the exact controllability of (1.2) in time  $L$  to the initial state  $z_0 = x_0$ .

We define  $\underline{z} \in C^0(\mathbb{R}; X)$  and  $\underline{v} \in L^2(\mathbb{R}; U)$  as the extensions of  $z$  and  $v$  by reflection with respect to  $s = 0$ , i.e.  $\underline{z}(s) = z(s) = \underline{z}(-s)$  and  $\underline{v}(s) = v(s) = \underline{v}(-s)$  for  $s \geq 0$ . They inherit from (2.4):

$$\underline{z}(t) = \cos(t)x_0 + \int_0^t \sin(t-s)B\underline{v}(s)ds. \quad (5.1)$$

Def. 2.2 of  $\kappa_{2,L}$  implies the following cost estimate for  $\underline{v}$ :

$$\int_0^L \|\underline{v}(s)\|^2 ds = 2 \int_0^L \|v(s)\|^2 ds \leq 2\kappa_{2,L}\|x_0\|^2. \quad (5.2)$$

Since  $D(A)$  is dense in  $X$ , there is a sequence  $(x_n)_{n \in \mathbb{N}^*}$  in  $D(A)$  converging to  $x_0$  in  $X$ . Since  $X_1$  is dense in  $X_{-1}$ , there is a sequence  $(f_n)_{n \in \mathbb{N}^*}$  in  $C^1(\mathbb{R}; X_1)$  converging to  $B\underline{v}$  in  $L^2(\mathbb{R}; X_{-1})$ . For each  $n \in \mathbb{N}^*$ , let  $\underline{z}_n$  be defined in  $C^2(\mathbb{R}; X)$  by:

$$\underline{z}_n(t) = \cos(t)x_n + \int_0^t \sin(t-s)f_n(s)ds,$$

which converges to  $\underline{z}(t)$  in  $X$  for all  $t$  due to (5.1). Since  $\underline{z}_n$  is a genuine solution of  $\ddot{\underline{z}}(t) = A\underline{z}_n(t) + f_n(t)$  (cf. lem. 4.1 of [14]), we have for all  $\varphi$  in  $D(A')$ :

$$s \mapsto \langle \underline{z}_n(s), \varphi \rangle \in H^2(\mathbb{R}) \quad \text{and} \quad \frac{d^2}{ds^2} \langle \underline{z}_n(s), \varphi \rangle = \langle \underline{z}_n(s), A'\varphi \rangle + \langle f_n(s), \varphi \rangle.$$

Hence,  $\langle \underline{z}_n(t), \varphi \rangle = \langle x_n, \varphi \rangle + \int_0^t (t-s) \langle \underline{z}_n(s), A'\varphi \rangle + \int_0^t (t-s) \langle f_n(s), \varphi \rangle$ . Passing to the limit, yields  $\langle \underline{z}(t), \varphi \rangle = \langle x_0, \varphi \rangle + \int_0^t (t-s) \langle \underline{z}(s), A'\varphi \rangle + \int_0^t (t-s) \langle B\underline{v}(s), \varphi \rangle$ . Therefore:

$$s \mapsto \langle \underline{z}(s), \varphi \rangle \in H^2(\mathbb{R}) \quad \text{and} \quad \frac{d^2}{ds^2} \langle \underline{z}(s), \varphi \rangle = \langle \underline{z}(s), A\varphi \rangle + \langle B\underline{v}(s), \varphi \rangle, \quad (5.3)$$

$$\langle \underline{z}(s), \varphi \rangle = 0 \quad \text{and} \quad \frac{d}{ds} \langle \underline{z}(s), \varphi \rangle = 0 \quad \text{for } |s| = L. \quad (5.4)$$

Let  $\alpha > \alpha_*$  and  $T \in ]0, \inf(1, L^2)[$ . Let  $\gamma > 0$  and  $k \in C^0([0, T]; \mathcal{M}(] - L, L[))$  be the corresponding constant and fundamental controlled solution given by proposition 4.2. We define  $\underline{k} \in C^0(\mathbb{R}_+; \mathcal{M}(\mathbb{R}))$  as the extension of  $k$  by zero, i.e.  $\underline{k}(t, s) =$



$\bar{k}(t, s)$  on  $[0, T] \times ]-L, L[$  and  $\underline{k}$  is zero everywhere else. It inherits from  $k$  the following properties

$$\partial_t \underline{k} = e^{i\theta} \partial_s^2 \underline{k} \quad \text{in } \mathcal{D}'([0, T] \times ]-L, L[), \quad (5.5)$$

$$\underline{k}|_{t=0} = \delta \quad \text{and} \quad \underline{k}|_{t=T} = 0, \quad (5.6)$$

$$\|\underline{k}\|_{L^2(\mathbb{R}_+ \times \mathbb{R})}^2 \leq \gamma e^{\alpha L^2/T}. \quad (5.7)$$

The main idea of the proof is to use  $\underline{k}$  as a kernel to transmute  $\underline{z}$  and  $\underline{v}$  into a solution  $x$  and a control  $u$  for (1.1). The transmutation formulas:

$$x(t) = \int \underline{k}(t, s) \underline{z}(s) ds \quad \text{and} \quad \forall t > 0, \quad u(t) = \int \underline{k}(t, s) \underline{v}(s) ds, \quad (5.8)$$

define  $x \in C^0(\mathbb{R}_+; X)$  and  $u \in L^2(\mathbb{R}_+; U)$  since  $\underline{k} \in C^0(\mathbb{R}_+; \mathcal{M}(\mathbb{R})) \cap L^2(\mathbb{R}_+; L^2(\mathbb{R}))$ ,  $\underline{z} \in C^0(\mathbb{R}; X)$  and  $\underline{v} \in L^2(\mathbb{R}; U)$ . The property (5.6) of  $\underline{k}$  implies  $x(0) = x_0$  and  $x(T) = 0$ . Equations (5.3), (5.4) and (5.5) imply, by integrating by parts:

$$\forall \varphi \in D(A'), \quad t \mapsto \langle x(t), \varphi \rangle \in H^1(\mathbb{R}_+) \quad \text{and} \quad \frac{d}{dt} \langle x(t), \varphi \rangle = \langle x(t), A' \varphi \rangle + \langle Bu(t), \varphi \rangle.$$

This characterizes  $x$  as the unique solution of (1.1) in the weak sense (cf. [5]), which implies that  $x$  and  $u$  satisfy (2.4). Since  $\int \|u(t)\|^2 dt \leq \iint |\underline{k}(t, s)|^2 ds dt \int \|\underline{v}(s)\|^2 ds$ , (5.7) and (5.2) imply the following cost estimate which completes the proof of th. 3.4:

$$\int_0^T \|u(t)\|^2 dt \leq 2\kappa_{2,L} \gamma e^{\alpha L^2/T} \|x_0\|^2.$$

**6. Geometric bounds on the cost of fast boundary controls for the heat equation.** When the second-order equation (1.2) has a finite propagation speed and is controllable, the control transmutation method yields geometric upper bounds on the cost of fast controls for the first-order equation (1.1). From this point of view, this method is an adaptation of the kernel estimates method of Cheeger-Gromov-Taylor in [10]. This was illustrated in [23] and [24] on the internal controllability of heat and Schrödinger equations on Riemannian manifolds which have the wave equation as corresponding second-order equation. Some similar lower bounds are proved in these papers (without assuming the controllability of the wave equation) which imply that the upper bounds are optimal with respect to time dependence. In this section, we illustrate the control transmutation method on the analogous boundary control problems for the heat equation.

Let  $(M, g)$  be a smooth connected compact  $n$ -dimensional Riemannian manifold with metric  $g$  and smooth boundary  $\partial M$ . When  $\partial M \neq \emptyset$ ,  $M$  denotes the interior and  $\bar{M} = M \cup \partial M$ . Let  $\Delta$  denote the (negative) Laplacian on  $(M, g)$  and  $\partial_\nu$  denote the exterior Neumann vector field on  $\partial M$ . The characteristic function of a set  $S$  is denoted by  $\chi_S$ .

Let  $X = L^2(M)$ . Let  $A$  be defined by  $Af = \Delta f$  on  $D(A) = H^2(M) \cap H_0^1(M)$ . Let  $C$  be defined from  $D(A)$  to  $U = L^2(\partial M)$  by  $Cf = \partial_\nu f|_\Gamma$  where  $\Gamma$  is an open subset of  $\partial M$ , and let  $B$  be the dual of  $C$ . With this setting, (1.1) with  $\theta = 0$  and (1.2) are the heat and wave equations controlled by the Dirichlet boundary condition on  $\Gamma$ . In particular (1.2) writes:

$$\begin{aligned} \partial_t^2 z - \Delta z &= 0 \quad \text{on } \mathbb{R}_t \times M, \quad z = \chi_\Gamma v \quad \text{on } \mathbb{R}_t \times \partial M, \\ z(0) &= z_0 \in L^2(M), \quad \dot{z}(0) = 0, \quad v \in L_{\text{loc}}^2(\mathbb{R}; L^2(\partial M)), \end{aligned} \quad (6.1)$$

It is well known that  $B$  is an admissible observation operator (cf. cor. 3.9 in [6]). To ensure existence of a null-control for the wave equation we use the geometrical optics condition of Bardos-Lebeau-Rauch (specifically example 1 after cor. 4.10 in [6]):

There is a positive constant  $L_\Gamma$  such that every generalized geodesic of length greater than  $L_\Gamma$  passes through  $\Gamma$  at a non-diffractive point. (6.2)

Generalized geodesics are the rays of geometrical optics (we refer to [21] for a presentation of this condition with a discussion of its significance). We make the additional assumption that they can be uniquely continued at the boundary  $\partial M$ . As in [6], to ensure this, we may assume either that  $\partial M$  has no contacts of infinite order with its tangents (e.g.  $\partial M = \emptyset$ ), or that  $g$  and  $\partial M$  are real analytic. For instance, we recall that (6.2) holds when  $\Gamma$  contains a closed hemisphere of a Euclidean ball  $M$  of diameter  $L_\Gamma/2$ , or when  $\Gamma = \partial M$  and  $M$  is a strictly convex bounded Euclidean set which does not contain any segment of length  $L_\Gamma$ .

**THEOREM 6.1** ([6]). *If (6.2) holds then the wave equation (6.1) is null-controllable in any time greater than  $L_\Gamma$ .*

Thanks to this theorem, th. 3.4 implies:

**THEOREM 6.2.** *If (6.2) holds then the equation:*

$$\begin{aligned} \partial_t x - e^{i\theta} \Delta x &= 0 \quad \text{on } \mathbb{R}_t \times M, \quad x = \chi_\Gamma u \quad \text{on } \mathbb{R}_t \times \partial M, \\ x(0) &= x_0 \in H^{-1}(M), \quad u \in L^2_{\text{loc}}(\mathbb{R}; L^2(\partial M)), \end{aligned}$$

is null-controllable in any time  $T$ . Moreover, the controllability cost  $\kappa_{1,T}$  (cf. def. 2.2) satisfies (with  $\alpha_*$  as in def. 3.2):  $\limsup_{T \rightarrow 0} T \ln \kappa_{1,T} \leq \alpha_* L_\Gamma^2$ .

**Remark 6.3.** Note that theorem 3.2 in [6] almost proves the converse of th. 6.1, thus showing that (6.2) is a sharp condition. Moreover [7, 8] prove that the full equivalence do hold when  $\chi_\Gamma$  is replaced by a smooth function with the same zero level set  $\partial\Omega \setminus \Gamma$  (this slight difference of model should not be perceptible in applications). Hence the control transmutation method does not allow to improve  $L_\Gamma$  in th. 6.2.

**Remark 6.4.** For more results and references on the cost of fast controls for the heat equation we refer to [15, 23]. It is a known fact that the heat equation is null-controllable for any non-empty control region  $\Gamma$ , hence the converse to the first statement of this theorem is not true. When  $M$  is a bounded Euclidean set and  $\Delta$  has constant coefficients, [15] proves that the limsup in the second statement is finite for any interior control region, but does not give an explicit bound. Motivated by the non-linear heat equation, it considers the heat equation with a time-dependent potential and describes how the cost depends on the norm of this potential, whereas the control transmutation method seems to apply only to time-independent potentials (on which the limsup above does not depend). As explained in section 2.4 of [23], in the general case without condition (6.2), one could try to adapt Carleman inequalities with phases  $\phi$  to obtain a bound in terms of the following distance function  $d : d(x, y) = \sup\{\phi(y) - \phi(x)\}$ , for all  $x$  and  $y$  in  $M$ , where the supremum is taken over all Lipschitz functions  $\phi : M \rightarrow \mathbb{R}$  with  $|\nabla \phi| \leq 1$  almost everywhere ( $d$  also has a geometric characterization in terms of path of least action).

**Remark 6.5.** This paper does not state explicitly a theorem about transmission problems for the linear heat equation because it does not address the main open problem in that context (cf. Open Problem §2.3.4 in [12]). Nonetheless, th. 3.4 provides a proof of some results which are not found in the literature (n.b. they are considered well-known by some experts). Firstly, it proves that “Russell’s principle” (cf. [27]) applies

to transmission problems in any dimension (this was explicitly assumed in §2.3.4 in [12] and proved in dimension 1 in [16], cf. footnote 3). Secondly, th. 3.4 deduces a new result on the boundary controllability of the transmission problems for the heat equation from the boundary controllability of the transmission problem for the wave equation in Tome 1, chap 6 in [20]. As in remark 6.4, since [12] is based on global Carleman estimates, it should extend from interior to boundary controls, and would yield fast controllability at an exponential rate under a less restrictive geometrical hypothesis than th. 3.4 does, but it would not yield an explicit bound on the rate as th. 3.4 does.

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