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Abstract. This paper proves that any initial condition in the energy space for the system of thermoelastic plates without rotatory inertia on a smooth bounded domain with hinged mechanical boundary conditions and Dirichlet thermal boundary condition can be steered to zero by a square integrable input function, either mechanical or thermal, supported in arbitrarily small sub-domain and time interval $[0,T]$. As T tends to zero, for initial states with unit energy norm, the norm of this input function grows at most like $\exp(C_p/T^p)$ for any real $p>1$ and some $C_p>0$. These results are analogous to the optimal ones known for the heat flow and the proof uses the heat control strategy of Lebeau and Robbiano.

1. Introduction

1.1. **Statement of the theorem.** We consider the system describing a linear thermoelastic plate of homogeneous material without rotatory inertia actuated by locally distributed inputs (cf. [LL88, Lag90], [Nor06] and references 11 to 42 therein):

$$\ddot{\zeta}+\Delta^2\zeta+\alpha\Delta\theta=\chi_\Omega u \quad \text{on } \mathbb{R}_+\times M, \tag{1a}$$

$$\dot{\theta}-\Delta\theta-\alpha\Delta\dot{\zeta}=\chi_\Omega v \quad \text{on } \mathbb{R}_+\times M, \tag{1b}$$

$$\zeta=\Delta\zeta=\theta=0 \quad \text{on } \mathbb{R}_+\times\partial M, \tag{1c}$$

where ζ is the vertical deflection, θ is the relative temperature about the stress free state $\theta=0$, α is a positive coupling parameter, u and v are respectively the mechanical and thermal input functions, M is a smooth connected bounded domain in $\mathbb{R}^d$, $\Delta=\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_d^2}$ denotes the Dirichlet Laplacian on $L^2(M)$ with domain $D(\Delta)=H_0^1(M)\cap H^2(M)$, χ_Ω denotes the multiplication by the characteristic function of the non-empty sub-domain $\Omega\subset M$. We consider square integrable input functions and initial data in the energy space:

$$\zeta=\zeta_0\in H^2(M)\cap H_0^1(M),\ \dot{\zeta}=\zeta_1\in L^2(M),\ \theta=\theta_0\in L^2(M),\quad \text{at } t=0. \tag{2}$$

The *controllability* property consists in the ability of steering any initial state $(\zeta_0,\zeta_1,\theta_0)$ to zero over a finite time by some appropriate mechanical input function u with $v=0$ or some thermal input function v with $u=0$ (this property is called null-controllability, as opposed to approximate controllability which is a weaker property not considered here, and exact controllability which is a stronger property never possessed by parabolic systems such as considered here). When the control time can be chosen as short as wished, we refer to this asymptotic as *fast control.* The *controllability cost* over a given time is the supremum over every initial state with unit energy norm of the smallest norm of a control (i.e. an input function) which steers it to zero over the given time. The blow-up of the cost of fast controls

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was first studied by Seidman (cf. ref. in [Sei05]) and recently connected by Da Prato to some properties of stochastic differential equations (cf. ref. in [AL03]).

Our controllability result for the system (1) of thermoelastic plates with either mechanical ($v = 0$) or thermal ($u = 0$) inputs is similar to the optimal fast controllability cost known for the heat flow, i.e. (1b) with $\alpha = 0$ (cf. [FCZ00]):

Theorem. *For all $\alpha > 0$, for all $\beta > 1$, there are $C_1 > 0$ and $C_2 > 0$ such that, for all $T \in (0, 1]$, for all initial data (2), there is an input function $u \in L^2((0, T) \times M)$ such that the solution of (1) with $v = 0$ satisfies $\zeta = \dot{\zeta} = \theta = 0$ at $t = T$ and the cost estimate:*

$$(3)\quad \int_0^T \int_M |u(t,x)|^2 dx\, dt \leqslant C_2 \exp\left(C_1/T^\beta\right) \int_M |\Delta\zeta_0(x)|^2 + |\zeta_1(x)|^2 + |\theta_0(x)|^2 dx \ .$$

The same statement holds with u and v exchanged.

1.2. **Background.** Avalos, Lasiecka and Triggiani proved in [Tri03, AL03] that for $\Omega = M$ (inputs distributed everywhere) the theorem holds with the cost $\exp\left(C_1/T^\beta\right)$ replaced by $1/T^5$ as if the system was finite-dimensional (cf. [Sei88]). The above theorem should still hold for $\beta = 1$ as for the heat flow (cf. [FCZ00]) and the system (1) with $u = v = 0$ in a cylindrical domain with thermal input on the whole base (cf. [Sei05] which announces this result of Lasiecka and Seidman). This exponential blow-up seems to be typical of infinite-dimensional systems with infinite propagation speed (heat, Schrödinger, plates, cf. [Sei05, Mil04]).

It seems that the above theorem is already new without the cost estimate. The controllability by thermal input in the theorem was proved by Benabdallah and Naso in [BN02] under an extra assumption on the input domain ($\overline{\Omega} \cap \partial M = \emptyset$). They also used the control strategy of Lebeau and Robbiano, but they neither considered $v = 0$, nor estimated the cost. Other controllability results concerned systems which reduce to one dimension (a beam controlled from the boundary in [HZ97], a rectangular plate controlled everywhere on one boundary edge in [Sei05]), or inputs distributed everywhere (cf. [Tri03, AL03]).

We should mention a widely studied version of (1) with the additional term $\gamma\Delta\ddot{\zeta}$ ($\gamma > 0$) which represents rotatory inertia (cf. [Lag90, HZ97] and ref. in [Nor06]):

$$(4a)\qquad \ddot{\zeta} - \gamma\Delta\ddot{\zeta} + \Delta^2\zeta + \alpha\Delta\theta = \chi_\Omega u \qquad \text{on } \mathbb{R}_+ \times M,$$

$$(4b)\qquad \dot{\theta} - \Delta\theta - \alpha\Delta\dot{\zeta} = \chi_\Omega v \qquad \text{on } \mathbb{R}_+ \times M,$$

$$(4c)\qquad \zeta = \Delta\zeta = \theta = 0 \qquad \text{on } \mathbb{R}_+ \times \partial M,$$

In (4a), $\gamma = 0$ corresponds to infinite speed of propagation and brings (4) back to the parabolic system (1) which corresponds to an analytic semigroup (cf. [LR95a]). On the contrary, $\gamma > 0$ corresponds to finite speed of propagation and (4) is a mixed hyperbolic-parabolic system which corresponds to a non-analytic semigroup.

This system (4) is very similar to the thermoelastic wave system studied in [LZ98]. Indeed Lebeau and Zuazua study essentially the system:

$$(5a)\qquad -\gamma\ddot{w} + \Delta w + \alpha\Delta\theta = \chi_\Omega u \qquad \text{on } \mathbb{R}_+ \times M,$$

$$(5b)\qquad \dot{\theta} - \Delta\theta - \alpha\dot{w} = \chi_\Omega v \qquad \text{on } \mathbb{R}_+ \times M,$$

$$(5c)\qquad w = \theta = 0 \qquad \text{on } \mathbb{R}_+ \times \partial M.$$

Setting $w = \Delta\zeta$, this system is almost the same as (4) since the additional term $\Delta^{-1}\ddot{w}$ in (4a) is a zero order perturbation of (5a). Therefore the method of [LZ98] should apply to (4). With this proviso, the "geometrical optics condition" on the control region Ω and the control time T is sufficient for the controllability of (4) with $u = 0$ or $v = 0$. N.b. it seems that much stronger geometrical assumptions on Ω are always made in the literature.

Earlier methods to estimate the cost of fast controls were based on sums of exponential functions (cf. [Sei05]), the transmutation control method (cf. [Mil04]), or global parabolic Carleman estimates (cf. [FCZ00]). We use a new method introduced in [Mil05], using the control strategy of Lebeau and Robbiano in [LR95a] as implemented in [LZ98] for the parabolic component of (5), ultimately based on local elliptic Carleman estimates.

It is clearly desirable to study thermoelastic plate equations with more realistic boundary conditions than (1c) or with locally distributed controls on the boundary instead of the interior. For systems with finite speed of propagation, controllability usually holds if and only if the control time T is greater than some critical positive time T_c, but there are no method yet for estimating the blow-up of the control cost as T tends to T_c. In particular, this is an open problem for the hyperbolic-parabolic system of thermoelastic plates (4).

2. Preliminaries

Before proving the theorem, we put it in the abstract semigroup framework, and introduce the main notations and ingredients.

2.1. **The duality between observation and control.** The proof of the theorem uses the well-known equivalence between controllability and observability (cf. [DR77]). In this section, we clarify in what sense the dual of the control problem (1) is the observation from Ω of the system (without input):

$$\ddot{z} + \Delta^2 z + \alpha\Delta\psi = 0 \quad \text{on } \mathbb{R}_+ \times M, \tag{6a}$$

$$\dot{\psi} - \Delta\psi - \alpha\Delta\dot{z} = 0 \quad \text{on } \mathbb{R}_+ \times M, \tag{6b}$$

$$z = \Delta z = \psi = 0 \quad \text{on } \mathbb{R}_+ \times \partial M, \tag{6c}$$

with initial data in the energy space:

$$z = z_0 \in H^2(M) \cap H_0^1(M),\ \dot{z} = z_1 \in L^2(M),\ \psi = \psi_0 \in L^2(M), \quad \text{at } t = 0. \tag{7}$$

The second-order differential systems (1) and (6) may be restated as first-order systems by setting $\eta(t) = (\Delta\zeta(t), \dot{\zeta}(t), \theta)$ and $y(t) = (\Delta z(t), -\dot{z}(t), \psi)$:

$$\dot{\eta}(t) = \mathcal{A}^*\eta(t) + \mathcal{B}_\Omega u(t), \qquad \eta(0) = \eta_0 \in Y, \quad u \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(M)), \tag{8}$$

$$\dot{y}(t) = \mathcal{A}y(t), \qquad y(0) = y_0 \in Y. \tag{9}$$

The state space $Y = L^2(M) \times L^2(M) \times L^2(M)$ is equipped with the energy norm $\|y\|^2 = \|y_1\|^2 + \|y_2\|^2 + \|y_3\|^2$. The semigroup generator $\mathcal{A}$ of (9) is defined by:

$$\mathcal{A} = \Delta A \text{ with } A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & \alpha \\ 0 & -\alpha & 1 \end{pmatrix}, \quad D(\mathcal{A}) = D(\Delta) \times D(\Delta) \times D(\Delta)\ . \tag{10}$$

The observation operator is the bounded operator from Y to $L^2(M)$ defined by $\mathcal{C}_\Omega = \chi_\Omega \Pi$, where Π denotes the projection from Y on the second (resp. third) coordinate for mechanical (resp. thermal) input. The control operator is its dual $\mathcal{B}_\Omega = \mathcal{C}_\Omega^*$, i.e. $\mathcal{B}_\Omega u = (0, \chi_\Omega u, 0)$ for mechanical input or $\mathcal{B}_\Omega u = (0, 0, \chi_\Omega u)$ for thermal input.

Proposition 1. *Let $T > 0$ and $C_T > 0$. The following properties are equivalent:*

(i) For all initial state $\eta_0 \in Y$, there is an input $u \in L^2_{loc}(\mathbb{R}; L^2(M))$ such that the solution $\eta \in C([0,\infty); Y)$ of (8) satisfies $\eta(T) = 0$ and $\|u\|_{L^2((0,T)\times M)} \leqslant C_T\|\eta_0\|$.

(ii) For all initial state $y_0 \in Y$, the solution $y(t) = e^{t\mathcal{A}}y_0$ of (9) satisfies the observation inequality: $\|y(T)\| \leqslant C_T\|\mathcal{C}_\Omega y(t)\|_{L^2((0,T)\times M)}$.

N.b. the smallest constant C_T such that these properties hold is the *controllability cost* mentioned in the introduction. The estimate in the theorem writes $C_T \leqslant C_2 \exp\left(C_1/T^\beta\right)$.

The proof of the exponential cost estimate in the theorem uses the polynomial cost estimate proved in [Tri03, AL03] in the simplest case $\Omega = M$. We restate this result of [Tri03, AL03] with our notations as the following observability inequality (due to exponential factors, it will be irrelevant here that $c_1 = 5$):

Proposition 2. *For all $\alpha > 0$, there are positive constants c_1 and c_2 such that for all $T \in (0,1]$ the solutions of (9) satisfy:*

$$\forall y_0 \in Y, \quad \|e^{T\mathcal{A}} y_0\|^2 \leqslant \frac{c_2}{T^{c_1}} \int_0^T \|\mathcal{C}_M e^{t\mathcal{A}} y_0\|^2 dt. \tag{11}$$

2.2. **Spectral decomposition.** As defined in the introduction, the operator Δ is a negative self-adjoint operator with compact resolvent. Let $(\omega_j)_{j\in\mathbb{N}^*}$ be a nondecreasing sequence of nonnegative real numbers and $(e_j)_{j\in\mathbb{N}^*}$ is an orthonormal basis of $L^2(M)$ such that e_j is an eigenvector of $-\Delta$ with eigenvalue ω_j^2, i.e.:

$$-\Delta e_j = \omega_j^2 e_j \quad \text{and} \quad e_j = 0 \text{ on } \partial M \ . \tag{12}$$

Let E_j denote the linear span of $(e_j, 0, 0)$, $(0, e_j, 0)$ and $(0, 0, e_j)$ in Y. For any frequency threshold μ, we introduce the following orthogonal decomposition of Y into low and high frequency spaces:

$$Y = E_{\leqslant\mu} \bigoplus E_{>\mu} \quad \text{with} \quad E_{\leqslant\mu} = \bigoplus_{\omega_j\leqslant\mu} E_j \quad \text{and} \quad E_{>\mu} = \bigoplus_{\omega_j>\mu} E_j \ . \tag{13}$$

A key ingredient in the proof of the theorem is the following inequality proved in [LZ98] (theorem 3) and [JL99] (theorem 14.6) by the local elliptic Carleman estimates in [LR95a]:

$$\exists C > 0, \forall \mu > 0, \forall v \in \mathbb{C}^{\mathbb{N}}, \quad \sum_{\omega_j\leqslant\mu} |v_j|^2 \leqslant Ce^{C\mu} \int_\Omega \left| \sum_{\omega_j\leqslant\mu} v_j e_j(x) \right|^2 dx \ .$$

We restate it as a “low-frequency observability inequality at exponential cost”:

Proposition 3. *There are positive constants D_0 and D_1, such that for all $\mu > 0$:*

$$\forall y \in E_{\leqslant\mu}, \quad \|\mathcal{C}_M y\| \leqslant D_0 e^{D_1\mu} \|\mathcal{C}_\Omega y\| \ . \tag{14}$$

Now we turn to the analysis of the spectrum $\sigma(A)$ of the matrix A defining $\mathcal{A}$ in (10). Since $\operatorname{Re} \bar{y}^T A y = |y_3|^2 \geqslant 0$ (energy decay) and $\alpha \neq 0$ implies that an eigenvector y of A has $y_3 \neq 0$, the spectral abscissa $\inf \operatorname{Re} \sigma(A)$ is positive.

We deduce from the orthogonal decomposition $y = \sum y_j$ in $\bigoplus E_j$, $\|y\|^2 = \sum \|y_j\|^2$ and $e^{t\mathcal{A}} y_j = e^{-t\omega_j^2 A} y_j$ that the semigroup generated by $\mathcal{A}$ leaves $E_{\leqslant\mu}$ and $E_{>\mu}$ invariant, inherits from the finite dimensional semigroup generated by A the property that its growth rate is determined by its spectral abscissa (it is even analytic, cf. [LR95b]), and thus satisfies the following “high-frequency exponential decay bound” with any rate $r \in (0, \min \operatorname{Re} \sigma(A))$:

Proposition 4. *For all $\alpha > 0$, there exists $r > 0$, such that for all $\mu > 0$:*

$$\forall y \in E_{>\mu}, \quad \forall t \geqslant 0, \quad \|e^{t\mathcal{A}} y\| \leqslant Ce^{-r\mu^2 t} \|y\| \ . \tag{15}$$

3. Proof of the theorem

Now the proof is essentially the same as in [Mil05] (where the generator was self-adjoint and c_1 was 1). For the sake of completeness, we customize it here. From the low-frequency stationary estimate in proposition 3, the observability from the whole domain M in proposition 2 and the duality in proposition 1, we deduce the "controllability at exponential cost" in the corresponding low-frequency dynamics. Combining it with the "high-frequency exponential decay bound" in proposition 4 according to the iterative control strategy introduced by Lebeau and Robbiano in [LR95a], we prove the controllability of all frequencies and estimate the controllability cost as the control time tends to zero.

Let $\tau \in (0,1]$, $\mu \geqslant 1$ and $y_0 \in E_{\leqslant\mu}$. For all $t \in [0,\tau]$, we may apply (14) to $e^{t\mathcal{A}}y_0$ since it is in $E_{\leqslant\mu}$: $\|\mathcal{C}_M e^{t\mathcal{A}}y_0\|^2 \leqslant D_0^2 e^{2D_1\mu}\|\mathcal{C}_\Omega e^{t\mathcal{A}}y_0\|^2$. Integrating on $[0,\tau]$ and using (11) yields: $\|e^{\tau\mathcal{A}}y_0\|^2 \leqslant D_0^2 e^{2D_1\mu} c_2 \tau^{-c_1}\int_0^\tau \|\mathcal{C}_\Omega e^{t\mathcal{A}}y_0\|^2 dt$. This is equivalent, by the same duality as in proposition 1, to the controllability property: for all $\tau \in (0,1]$ and $\mu > 1$, there is a bounded operator $S_\mu^\tau : Y \to L^2(0,\tau;L^2(M))$ such that, for all $\eta_0 \in E_{\leqslant\mu}$, the solution $\eta \in C([0,\infty),Y)$ of (8) with input function $u = S_\mu^\tau \eta_0$ satisfies $\eta(\tau) \in E_{>\mu}$, and $\|S_\mu^\tau\| \leqslant \frac{D_2}{\tau^{c_1/2}} e^{D_1\mu}$ with $D_2 = D_0\sqrt{c_2} > 0$.

We introduce a dyadic scale of modes $\mu_k = 2^k$ $(k \in \mathbb{N})$ and a sequence of time intervals $\tau_k = \sigma_\delta T/\mu_k^\delta$ where $\delta \in (0,1)$ and $\sigma_\delta = (2\sum_{k\in\mathbb{N}} 2^{-k\delta})^{-1} > 0$, so that the sequence of times defined recursively by $T_0 = 0$ and $T_{k+1} = T_k + 2\tau_k$ converges to T. The strategy consists in steering the initial state η_0 to 0 through the sequence of states $\eta_k = \eta(T_k) \in E_{>\mu_{k-1}}$ at frequencies converging to infinity by applying recursively the input function $u_k = S_{\mu_k}^{\tau_k}\eta_k$ to η_k during a time τ_k and no input during a time τ_k. A byproduct of later estimates is that η_k converges to zero.

To estimate the cost C_T formally defined after proposition 1, we introduce:

$$\varepsilon_k = \|\eta_k\|, \quad C_k = D_2 e^{D_1\mu_k}/\tau_k^{c_1/2} \quad \text{and} \quad \rho_k = \left(\frac{C_{k+1}\varepsilon_{k+1}}{C_k\varepsilon_k}\right)^2 . \tag{16}$$

With these notations, the iteration cost satisfies $\|S_{\mu_k}^{\tau_k}\| \leqslant C_k$, the full input function u satisfies $\int_0^T \|u\|^2 dt = \sum_k \int_0^{\tau_k} \|u_k\|^2 dt \leqslant \sum_{k\in\mathbb{N}} C_k^2 \varepsilon_k^2$ and the full cost satisfies:

$$C_T^2 \leqslant C_0^2 \left(1 + \sum_{l\geqslant 1} \prod_{0\leqslant k\leqslant l-1} \rho_k\right). \tag{17}$$

Since $\mathcal{B}_\Omega$ and $e^{t\mathcal{A}}$ are both bounded by 1, the integral formula expressing the final state $\eta(T_k+\tau_k)$ in terms of the initial state $\eta(T_k)$ and the source term $\mathcal{B}_\Omega u_k$ implies $\|\eta(T_k+\tau_k)\|^2 \leqslant 2\|e^{\tau_k\mathcal{A}}\eta(T_k)\|^2 + 2\tau_k\|u_k\|^2_{L^2(0,\tau_k;L^2(M))} \leqslant 2(1+C_k^2)\varepsilon_k^2$. Since $\varepsilon_{k+1} \leqslant Ce^{-r\mu_k^2\tau_k}\|\eta(T_k+\tau_k)\|$ by applying (15) to $\eta(T_k+\tau_k) \in E_{>\mu_k}$, we deduce: $\varepsilon_{k+1}^2 \leqslant 2Ce^{-2r\tau_k\mu_k^2}(1+C_k^2)\varepsilon_k^2$. Since $C_{k+1}/C_k = 2^{\delta c_1/2}e^{D_1\mu_k}$, we deduce that, for any $D_3 > 4D_1$, there is a $D_4 > 0$ such that:

$$\rho_k \leqslant C2^{1+\delta c_1}\left(e^{-2D_1\mu_k} + \frac{D_2^2}{\tau_k^{c_1}}\right)e^{4D_1\mu_k - 2r\tau_k\mu_k^2} \leqslant \frac{D_4}{T^{c_1}} e^{D_3\mu_k - 2r\sigma_\delta T\mu_k^{2-\delta}} .$$

Since $l \leqslant \mu_l$, $\sum_{0\leqslant k\leqslant l-1}\mu_k \leqslant \mu_l$ and $\sum_{0\leqslant k\leqslant l-1}\mu_k^{2-\delta} \geqslant \mu_{l-1}^{2-\delta}/2$, this implies $\prod_{0\leqslant k\leqslant l-1}\rho_k \leqslant \exp\left((D_3 + \ln(D_4/T^{c_1}))\,\mu_l - r\sigma_\delta T\mu_{l-1}^{2-\delta}\right)$. Hence, setting $q = 2^{2-\delta}$ and $T' = r\sigma_\delta T/q$: $\prod_{0\leqslant k\leqslant l-1}\rho_k \leqslant \exp\left(D_{T'}2^l - T'q^l\right)$ with $D_{T'} \underset{T'\to 0}{\sim} c_1\ln(1/T')$. Using (17) and setting $D_5 = D_2^2 e^{2D_1} r^{c_1}/q^{c_1}$, we deduce the cost estimate:

$$C_T^2 \leqslant \frac{D_5}{T'^{c_1}}\left(1 + \sum_{k\geqslant 1}\exp\left(D_{T'}2^k - T'q^k\right)\right) \text{ with } D_{T'} \underset{T'\to 0}{\sim} c_1\ln(1/T') . \tag{18}$$

To estimate the last sum, we shall use the simple estimate:

$$f(t) := \sum_{k\geqslant 1} \exp\left(-tq^k\right) \leqslant \sum_{k\geqslant 1} \exp\left(-tk\right) = \frac{e^{-t}}{1-e^{-t}} \leqslant \frac{1}{t} \ . \tag{19}$$

Let $\varepsilon \in (0,1)$ and $h_\varepsilon(x) = D_{T'}2^x - \varepsilon T' q^x$. The maximum of the function h_ε on $\mathbb{R}$ is obtained at a point x_ε which satisfies, since $D_{T'} \underset{T'\to 0}{\sim} c_1 \ln(1/T')$:

$$x_\varepsilon = \frac{\ln\left(D_{T'} \ln 2/(\varepsilon T' \ln q)\right)}{\ln(q/2)} \underset{T'\to 0}{\sim} \frac{\ln(1/T')}{\ln(q/2)} = \frac{1+\beta_q}{\ln q} \ln(1/T') \ ,$$

where $\beta_q = \left(\frac{\ln q}{\ln 2} - 1\right)^{-1}$. Therefore, $\forall \beta > \beta_q$, $\exists T_\beta > 0$, $\forall T' \in (0, T_\beta)$:

$$x_\varepsilon \ln q \leqslant (1+\beta)\ln(1/T') \quad \text{, hence} \quad h_\varepsilon(x_\varepsilon) = \frac{\varepsilon T'}{\beta_q} q^{x_\varepsilon} \leqslant \frac{\varepsilon}{\beta_q T'^\beta} \ .$$

The inequality $h_\varepsilon(x) \leqslant h_\varepsilon(x_\varepsilon)$ implies $D_{T'}2^x - T'q^x \leqslant h_\varepsilon(x_\varepsilon) - (1-\varepsilon)T'q^x$. Using this with $x = k$ for $k \geqslant 1$, the above bound on $h_\varepsilon(x_\varepsilon)$ and (19) yield:

$$\sum_{k\geqslant 1} \exp\left(D_{T'}2^k - T'q^k\right) \leqslant e^{h_\varepsilon(x_\varepsilon)} f((1-\varepsilon)T') \leqslant \exp\left(\frac{\varepsilon}{\beta_q T'^\beta}\right) \frac{1}{(1-\varepsilon)T'} \ .$$

Plugging this in (18) yields: $\forall \beta > \beta_q$, $\exists D_6 > 0$, $\exists D_7 > 0$, $C_T^2 \leqslant D_6 \exp\left(D_7/T'^\beta\right)$. Since T' is proportional to T and β_q decreases to 1 as δ decreases to 0, this proves the cost estimate in the theorem as restated after proposition 1.

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