Unique continuation estimates for sums of semiclassical eigenfunctions and null-controllability from cones

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UNIQUE CONTINUATION ESTIMATES FOR SUMS OF SEMICLASSICAL EIGENFUNCTIONS AND NULL-CONTROLLABILITY FROM CONES

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Abstract. For all sums of eigenfunctions of a semiclassical Schrödinger operator below some given energy level, this paper proves that the ratio of the L^2 norm on $\mathbb{R}^d$ over the L^2 norm on any given open set is bounded by $\exp(C/h)$ for some positive C in the semiclassical limit h tends to 0. Corresponding estimates on a compact manifold are also given. They generalize the unique continuation estimate of Lebeau, with Jerison, Robbiano and Zuazua, on sums of classical eigenfunctions of the Laplacian on a compact manifold below an eigenvalue threshold as this threshold tends to infinity. The main tools are semiclassical Carleman estimates following Lebeau, Robbiano and Burq with a new semiclassical propagation of smallness argument.

For sums of classical Hermite functions, or for sums of classical eigenfunctions of homogeneous polynomial potential wells, similar unique continuation estimates from cones are deduced. They apply to the null-controllability from a cone of the heat semigroups corresponding to these Schrödinger operators, with a sharp cost estimate of fast control, following a new version of the strategy of Lebeau and Robbiano.

1. Introduction

1.1. **Main results.** We are interested in sums of eigenfunctions of some self-adjoint operators A on a Hilbert space L^2 bounded from below with compact resolvent. The space generated by the eigenfunctions with eigenvalues lower than the real number E will be denoted[1] by $\mathbf{1}_{A<E}\, L^2$ (n.b. its elements are sums of eigenfunctions).

We consider a power $k \in (0,+\infty)$ and a potential well $V \in \mathcal{C}^\infty(\mathbb{R}^d)$ which behaves like $|x|^{2k}$ at infinity. More precisely, for some $R>0$ and all multi-index α,

$$(1) \qquad |\partial^\alpha V(x)| \leqslant C_\alpha (1+|x|^2)^k \quad \text{and} \quad |x| \geqslant R \Rightarrow V(x) \geqslant C_R(1+|x|^2)^k \ .$$

The semiclassical Schrödinger operator $P = -h^2\Delta + V(x)$ with domain $D(P) = \left\{ u \in H^2(\mathbb{R}^d) \mid \int |Vu|^2 < \infty \right\}$ is self-adjoint, bounded from below and has compact resolvent. Indeed we allow P to be a more general semiclassical elliptic operator with second order coefficients and h-dependent lower order terms, cf. section 2.

Theorem 1.1. *For any non empty open subset Ω of $\mathbb{R}^d$ and any energy E_0, there are positive constants h_0 and C_0 such that*

$$(2) \qquad \forall h \in (0,h_0), \forall u \in \mathbf{1}_{P<E_0}\, L^2(\mathbb{R}^d), \int_{\mathbb{R}^d} |u(x)|^2 dx \leqslant e^{C_0/h} \int_\Omega |u(x)|^2 dx \ .$$

Another consequence of the proof of this theorem is the following unique continuation estimate which generalizes the lower bound for eigenfunctions proved in

Date: November 4, 2009. The first version was submitted on November 21, 2008.

2000 *Mathematics Subject Classification.* 35B37, 35B60, 35P20, 93B07.

[1] To denote the spectral projector on this space by $\mathbf{1}_{A<E}$ is consistent with the functional calculus for self-adjoint operators applied to the characteristic function $\mathbf{1}_{\cdot<E}$ defined on $\mathbb{R}$ by $\mathbf{1}_{\lambda<E} = 1$ if $\lambda < E$ and $\mathbf{1}_{\lambda<E} = 0$ otherwise.

[EZ07, section 7.2] (n.b. although theorems 1.1 and 1.2 give the same estimate of $\int_{\mathbb{R}^d}|u(x)|^2dx$ when $Pu = Eu$, $|E| < E_0$, neither follows from the other):

Theorem 1.2. *For any non empty open $\Omega \subset \mathbb{R}^d$ and any energy E_0, there are positive constants h_0 and C_0 such that for all $h \in (0, h_0)$ and $E(h) \in (-E_0, E_0)$,*

$$\forall u \in \mathrm{D}(P), \int_{\mathbb{R}^d}|h\nabla u|^2 + (1+|x|^2)^k|u|^2 \leqslant e^{C_0/h}\left(\int_\Omega |u|^2 + \int_{\mathbb{R}^d}|(P-E(h))u|^2\right).$$

Theorem 1.1 also holds on a smooth connected compact d-dimensional Riemannian manifold M with metric g and boundary ∂M. Let Δ_g denote the Laplace-Beltrami operator (n.b. Δ_g is a negative differential operator with variable coefficients depending on the metric g).

Theorem 1.3. *Let $V \in C^\infty(M;\mathbb{R})$. Let $P = -h^2\Delta_g + V(x)$ denote the operator on $L^2(M)$ with domain $\mathrm{D}(P) = H_0^1(M) \cap H^2(M)$. For any non empty open subset Ω of M and any energy E_0, there are positive constants h_0 and C_0 such that*

$$\forall h \in (0, h_0), \forall u \in \mathbf{1}_{P<E_0} L^2(M), \int_M |u(x)|^2dx \leqslant e^{C_0/h}\int_\Omega |u(x)|^2dx\ . \tag{3}$$

Remark 1.4. The method used to prove theorem 1.3 in section 3.3 also yields theorem 1.2 for $P = -h^2\Delta_g + V - E(h)$ on a compact manifold with smooth V :

$$\forall u \in H_0^1(M) \cap H^2(M), \int_M |h\nabla u|^2 + |u|^2 \leqslant C(h)\left(\int_\Omega |u|^2 + \frac{1}{h^2}\int_M |Pu|^2\right), \tag{4}$$

with $C(h) = e^{C_0/h}$ (when $V = 0$, it also yields the corresponding corollary 1.7 with exponent 1 instead of $1 + \frac{1}{k}$). It also yields the same estimate with the integral of $|u|^2$ over Ω replaced by the integral of the Neumann derivative $|h\partial_N u|^2$ over a given subset of ∂M. N.b. the factor $1/h^2$ is meaningless as long as $C(h)$ is exponential and was only kept for the sake of the following discussion in remark 1.5.

The power of h in the factor $e^{C_0/h}$ in theorems 1.1, 1.2 and 1.3, and in (4) is already optimal for single eigenfunctions for some Ω. When $\overline{\Omega}$ is a compact set in the classically forbidden region $V^{-1}(E_0, +\infty)$ this results from semiclassical Agmon estimates (cf. e.g. [DS99, HN05, EZ07]). It results from section 4.2.2 that this is still valid for some larger Ω in the following two cases: (i) when $V(x) = |x|^{2k}$, $k \in \mathbb{N}^*$, in theorems 1.1 and 1.2, for any cone Ω as in remark 1.9; (ii) when M is a sphere and $V = 0$ in (3) and (4), for any neighborhood $M \setminus \Omega$ of a great circle.

When $V = 0$ in theorem 1.3 for *sums* of eigenfunctions the power of h in $e^{C_0/h}$ is optimal for *any* $\Omega \neq M$ in contrast to remark 1.5 (this already results from [JL99, proposition 14.9]). Moreover, in this case the rate in $e^{C_0/h}$ was recently bounded from below in [Mil09, theorem 5.1]: $C_0 \geqslant \sup_{y\in M}\mathrm{dist}(y, \overline{\Omega})$.

Remark 1.5. We recall that (4) for $P = -h^2\Delta_g - 1$ on a compact manifold (and therefore (5) for single eigenfunctions instead of sums) is valid with $C(h)$ independent of h (i.e. without the exponential factor) in the following three cases: (i) under the condition of Bardos-Lebeau-Rauch that all generalized geodesics intersect Ω; (ii) if the integral of $|u|^2$ over Ω is replaced by the integral of the Neumann derivative $|h\partial_N u|^2$ over a given subset of ∂M intersected by all generalized geodesics; (iii) if M is a partially rectangular billard under some condition on Ω weaker than in (i), cf. e.g. [Bur02b], [BZ04, (6.8),(6.14)], [Mar06]. It is also valid with a logarithmic factor $C(h)$ instead of exponential (cf. [Bur04, BZ04, Chr07] for more details) in the following two cases: (i) if $M \subset \mathbb{R}^d$ has convex holes which are far away from each other as in Ikawa's scattering result; (ii) if $M \setminus \Omega$ is a neighborhood of a hyperbolic closed geodesic.

We refer to [KT05, KTZ07] for recent related L^p estimates of eigenfunctions.

1.2. **Background and non semiclassical consequences.** Taking $V = 0$, $E_0 = 1$ and $h = 1/\mu$ in theorem 1.3 yields :

$$\forall \mu \geqslant 0, \forall v \in \mathbf{1}_{-\Delta_g < \mu^2} L^2(M), \int_M |v(x)|^2 dx \leqslant C e^{C\mu} \int_\Omega |v(x)|^2 dx \ . \tag{5}$$

This estimate was proved in [LZ98, theorem 3] and [JL99, theorem 14.6] by the semiclassical local elliptic Carleman estimates proved in [LR95]. It has been generalized to a piecewise smooth g discontinuous across a smooth interface in [LRR]. We refer to [BHLR09] for discrete versions and to the recent survey [LRL09]. It is already non trivial for trigonometric sums (eigenfunctions of the Dirichlet Laplacian on a segment or a circle).

Similarly, the following corollary with $k = 1$ is already a non trivial result on sums of Hermite functions (eigenfunctions of the harmonic oscillator). By the scaling in section 4.1, theorem 1.1 with the homogeneous polynomial and radial potential well $V(x) = |x|^{2k}$, $k \in \mathbb{N}^*$, yields the following *unique continuation estimate from cones* (a.k.a. observability estimate of sums of eigenfunctions from Γ) :

Corollary 1.6. *Let $A = -\Delta + |x|^{2k}$ with $k \in \mathbb{N}^*$. For any non empty open cone $\Gamma = \{x \in \mathbb{R}^d \mid |x| > r_0, x/|x| \in \Omega_0\}$, where $r_0 \geqslant 0$ and Ω_0 is an open subset of the unit sphere, there is a $C > 0$ such that*

$$\forall \mu \geqslant 0, \forall v \in \mathbf{1}_{A<\mu^2} L^2(\mathbb{R}^d), \int_{\mathbb{R}^d} |v(x)|^2 dx \leqslant C e^{C\mu^{1+1/k}} \int_\Gamma |v(x)|^2 dx \ . \tag{6}$$

By the same scaling, the following *observability resolvent estimate* (remark 1.13 gives some background) is deduced from theorem 2.6 in section 4.1:

Corollary 1.7. *Let $A = -\Delta + |x|^{2k}$ with $k \in \mathbb{N}^*$. For any non empty open cone Γ as in corollary 1.6, there is a $C > 0$ such that, for all $\lambda \in \mathbb{C}$ and $v \in \mathrm{D}(A)$,*

$$\int_{\mathbb{R}^d} |\nabla v|^2 + (1 + |x|^2)^k |v|^2 \leqslant C e^{C(\sqrt{\lambda})^{1+1/k}} \left(\int_\Gamma |v|^2 + \int_{\mathbb{R}^d} |(A - \lambda)v|^2 \right) \ , \tag{7}$$

where the square root is defined as $\sqrt{\lambda} = (\max\{\operatorname{Re}\lambda, 0\})^{1/2}$.

Remark 1.8. Both corollaries still hold if $|x|^{2k}$ is replaced by any smooth potential well V which satisfies (1) and is positively homogeneous of degree $2k$, cf. (24).

Remark 1.9. Although the potential well "compactifies infinity", note the loss of $\frac{1}{k}$ in the power of μ when comparing (6) to (5). It is proved in section 4.2 that the powers $1 + \frac{1}{k}$ in corollaries 1.6 and 1.7 are both sharp even for single eigenfunctions instead of sums, and even if ∇ and $|x|$ are omitted in (7), at least when $d \geqslant 3$ and there is a vector space Π of dimension $d - 1$ such that $\overline{\Gamma} \cap \Pi \subset \{0\}$ (more precisely, if there is a vector space of dimension 2 in $\mathbb{R}^d$ not intersecting $\overline{\Omega_0}$). Moreover, when $k = 1$, the power in corollary 1.6 is sharp for any d and Γ due to theorem 1.10.

When Γ is a bounded set instead of a cone, it is also proved in section 4.2 that both corollaries fail even for single eigenfunctions instead of sums by at least an extra logarithmic factor in the exponentials. Whether the estimate (6) holds with $\mu^{1+\frac{1}{k}}$ replaced by $\mu^{1+\frac{1}{k}} \ln \mu$ when Γ is bounded remains open.

1.3. **Null-controllability of parabolic semigroups in unbounded domains.** Recall $k \in \mathbb{N}^*$, $V(x) = |x|^{2k}$, $A = -\Delta + V$, $D(A) = \{u \in H^2(\mathbb{R}^d) \mid \int |Vu|^2 < \infty\}$. Let χ denote the multiplication by the characteristic function of some set Γ in R^d. Consider the parabolic equation with input function $u \in L^2([0,T]; L^2(\mathbb{R}^d))$

$$\partial_t \phi - \Delta\phi + V\phi = \chi u, \quad \text{with initial state } \phi(0) = \phi_0 \in L^2(\mathbb{R}^d) \ . \tag{8}$$

Null-controllability at time T holds if for all ϕ_0 there is a u such that the solution $\phi \in \mathcal{C}([0,T]; L^2(\mathbb{R}^d))$ of (8) satisfies $\phi(T) = 0$. This property is equivalent to the following unique continuation estimate known as *final-observability* at time T:

$$\exists \kappa_T > 0, \forall v \in L^2(\mathbb{R}^d), \|e^{-TA}v\|^2 \leqslant \kappa_T \int_0^T \|\chi e^{-tA}v\|^2 dt \; . \tag{9}$$

Theorem 1.10. *Let Γ be any open cone as in corollary 1.6.*

The potential well (with power twice $k \in \mathbb{N}^*$) satisfies $k > 1$ if and only if null-controllability holds at all times T. Moreover the controllability cost κ_T in (9) *satisfies: $\kappa = \limsup_{T\to 0} T^\beta \ln \kappa_T < \infty$ with $\beta = 1 + 2/(k-1)$.*

If there is a vector space of dimension 2 in $\mathbb{R}^d$ which does not intersect the closure $\overline{\Omega}_0$ of the subset Ω_0 of the unit sphere defining the cone Γ then $\kappa \neq 0$.

If $k = 1$ (harmonic oscillator) and Γ is (inside) a half-space then null-controllability does not hold at any time T.

The negative part of theorem 1.10 is proved in section 4.3. Corollary 1.6 and [Mil06, theorem 1] prove (even in the setting of remark 1.8) the positive part of theorem 1.10, but only prove the upper bound of κ for any exponent greater than β. The cost properties of κ stated in theorem 1.10, i.e. the exponent β is valid and sharp, were deduced from corollary 1.6 and section 4.2.2 recently in [Mil09] (when Γ is a bounded set instead of a cone, $\kappa = +\infty$ was deduced from section 4.2.3).

Remark 1.11. These null-controllability results in $\mathbb{R}^d$ complement those obtained for parabolic equations on various unbounded domain [MZ01, CDMZ01, Mil05c, GBdT07] by saying roughly that, although null-controllability from cones Γ does not hold with the quadratic potential well $V(x) = |x|^2$ or in the flat case $V = 0$ (cf. [Mil05b]), it does hold for more confining potentials $V(x) = |x|^{2k}$ with $k > 1$. Whether null-controllability from bounded sets Γ holds for $k > 1$ remains open.

We refer to [Èma95, FCZ00, VZ08] for null-controllability results for the heat semigroup with potentials on bounded domains motivated by nonlinear problems.

Remark 1.12. The criterion $k > 1$ in theorem 1.10 is reminiscent of the one found for the *contractivity properties* of the semigroup e^{-tA} ([DS84]): it is intrinsically ultracontractive if $k > 1$, intrinsically hypercontractive but not supercontractive if $k = 1$, and not even intrinsically hypercontractive if $k < 1$. But we would like to point out that there are no direct proofs of final-observability for a parabolic semigroup based on its kernel even when the kernel is explicitly known.

Although not used here, we recall that for $k > 1$ this contractivity property together with the ground state estimate $\exists C_0 > 0$, $\forall x \in \mathbb{R}^d$, $C_0^{-1}\phi_0(x) \leqslant (1 + |x|)^{-(k+d-1)/2} \exp\left(-|x|^{1+k}/(1+k)\right) \leqslant C_0\phi_0(x)$ ([Dav89, Corollary 4.5.8]), entail the properties: *decay of eigenfunctions* ([Dav89, theorem 4.2.4]) $A\phi = \lambda\phi \Rightarrow \exists C_\lambda > 0$, $|\phi| \leqslant C_\lambda \phi_0$; *kernel estimate for large time* ([Dav89, theorem 4.2.5]) $\forall \varepsilon > 0$, $\exists T > 0$, $t > T \Rightarrow (1-\varepsilon)e^{-t\lambda_0}\phi_0(x)\phi_0(y) \leqslant e^{-tA}(x,y) \leqslant (1+\varepsilon)e^{-t\lambda_0}\phi_0(x)\phi_0(y)$; *kernel estimate for small time* ([Dav89, corollary 4.5.5] and [MS07] for the limit exponent) $\exists C > 0, \forall x, y \in \mathbb{R}^d, 0 < t \leqslant 1 \Rightarrow 0 \leqslant e^{-tA}(x,y) \leqslant C \exp\left(C/t^{1+2/(k-1)}\right)\phi_0(x)\phi_0(y)$.

Remark 1.13. When Γ is a cone for which corollary 1.7 is sharp as in remark 1.9, the Hautus test [Mil05a, theorem 5.1] for exact controllability of the Schrödinger group generated by iA prove that *exact controllability of the Schrödinger equation* $i\partial_t\phi - \Delta\phi + V\phi = \chi u$ does not hold at any time T, which means (due to the group property) : $\forall T$, $\forall \phi_T \in L^2(\mathbb{R}^d)$, $\exists \phi_0 \in L^2(\mathbb{R}^d)$, $\forall u \in L^2([0,T]; L^2(\mathbb{R}^d))$, $\phi(0) = \phi_0 \Rightarrow \phi(T) \neq \phi_T$.

Omitting ∇ and $|x|$ in the observability resolvent estimate (7) yields

$$\forall \lambda \in \mathbb{C}, \forall v \in \mathrm{D}(A), \quad \|v\|^2 \leqslant Ce^{C(\sqrt{\lambda})^\alpha}\left(\|\chi v\|^2 + \|(A-\lambda)v\|^2\right) \; , \tag{10}$$

with $\alpha = 1 + 1/k$. This still makes sense when A is any positive self-adjoint operator on a Hilbert space X and χ is any bounded operator from X to another Hilbert space Y. One may wonder under which additional assumption (10) with $\alpha \in (0, 2)$ implies the null-controllability of the semigroup generated by $-A$. About the converse, Thomas Duyckaerts noted that null-controllability at time T implies (10) with $\alpha = 2$ (this can be proved by mainly changing i into -1 in [Mil05a, lemma 5.2]), cf. [DM09].

1.4. **Main ideas, plan and notations.** The main results are proved in sections 2 and 3 by a fully semiclassical version of the strategy introduced in [LZ98, JL99] for sums of classical eigenfunctions. The main tools are semiclassical Carleman estimates in the spirit of [LR95, Bur98, Bur02a] with three new features. The first feature deals with ellipticity at infinity by building a global phase function which fulfills Hörmander's hypoellipticity condition in an *unbounded* region (the exterior of a ball, cf. sections 2.1 and 2.2). We first use this phase in section 2.3 to prove a global unique continuation estimate for P. We also use it in section 3.1 to prove a similar continuation estimate from the boundary $\{t = 0\}$ for $h^2\partial_t^2 + P$ where t appears as an artificial elliptic variable following the strategy of [LZ98, JL99].

The second feature could be be coined *semiclassical propagation of smallness*, i.e. combining a cascade of semiclassical unique continuation estimates into a single more global one. Whereas Lebeau and Robbiano first optimize these semiclassical estimates with respect to h to obtain interpolation inequalities and then propagate, we show how to propagate these semiclassical estimates directly by choosing each new Carleman phase steep enough with respect to the previous one (taking advantage of the "convexity parameter" β, cf. sections 3.2 and 3.3). The last feature simplifies the strategy of [LZ98, JL99] in order to keep the variable h in the final results (for later scaling in section 4.1). Indeed we circumvent the interpolation inequalities altogether (cf. section 3.1). Another way to circumvent interpolation inequalities to prove (5) when M is a bounded domain is introduced in [LR07, sect. 3.A] and [BHLR09], based on a global Carleman estimate from the boundary for $\partial_t^2 + \Delta_g$. Whether such elliptic or directly parabolic global Carleman estimates in the spirit of [Èma95] allow to generalize theorem 1.1 and corollary 1.6 to a wider class of potentials and to semilinear parabolic equations remains open.

The non semiclassical results are all proved in section 4 under the stronger hypothesis on the potential well that it is a homogeneous polynomial. This allows scaling in section 4.1. Sharpness of the exponents in the observability estimates are proved by exhibiting some radial eigenfunctions: eigenfunctions concentrating at some "equator" saturate the estimates, semiclassical Agmon estimates with respect to the radial coordinate refute the estimates when observing from bounded sets. Non-controllability for $k = 1$ is proved using the explicit Mehler formula for the Hermite kernel.

Unless mentioned otherwise, the norm $\|\cdot\|$ and the hermitian scalar product $\langle \cdot, \cdot \rangle$ are in L^2. The Poisson bracket is denoted $\{\cdot, \cdot\}$ and the operator commutation bracket is denoted $[\cdot, \cdot]$. The transpose of a vector ξ is denoted ${}^t\xi$. The notation $\mathcal{U} \Subset \mathcal{H}$ means $\mathcal{U}$ is an open set such that its closure $\overline{\mathcal{U}}$ is included in the set $\mathcal{H}$.

Acknowledgement. This work was partially supported by Grant JCJC06-137283 of the Agence Nationale de la Recherche. I am grateful to Jérôme Le Rousseau and Luc Robbiano for pointing out the related independent works [LR07, sect. 3.A] and [Van09] as they browsed the first draft of this paper. The already submitted paper of Michael VanValkenburgh considers theorem 1.3 on $M \subset \mathbb{R}^d$ with a Riemannian metric for semiclassical quasimodes instead of sums of eigenfunctions (i.e. u is an eigenfunction up to an exponentially small error with respect to h). In this context,

(4) is a slightly more general statement of its results. In order to piece together two Carleman estimates, it uses two weights with a certain compatibility condition following [Bur98].

2. Semiclassical Carleman estimates

The results and notations for semiclassical analysis used in this section can be found in [DS99] (cf. [EZ07, section 7.2] for a nice introduction to semiclassical Carleman estimates). In this context it is more natural to prove theorem 1.1 for the following more general Schrödinger operators P.

In this section, $P = p(x, hD, h)$ is a semiclassical differential operator with semiclassical symbol $p(x, \xi, h) = q(x, \xi) + l(x, h).\xi + v(x, h) + V(x)$, where V satisfies (1), q is a uniformly positive quadratic form given by $q(x, \xi) = {}^t\xi A(x)\xi \geqslant q_0|\xi|^2$, all the derivatives with respect to x of the real valued functions A, l, v are uniformly bounded with respect to $(x, h) \in \mathbb{R}^d \times (0, h_0]$. From now on, the notations won't mention the fact that p may depend on h thus p is a principal symbol (N.b. the setting in [Bur02a] is similar with this major discrepancy: P is a long range perturbation of the semiclassical Laplace operator, i.e. $V = 0$ and $A - Id$, l and v are $O(|x|^{-\alpha})$ as $|x| \to \infty$ for some $\alpha > 0$.)

As usual in the proof of Carleman estimates, we introduce the conjugate of P with a real smooth phase function φ (the multiplication by the function $e^{\varphi/h}$ is denoted by the same symbol):

$$P_\varphi := e^{\varphi/h} \circ P \circ e^{-\varphi/h} \text{ with principal symbol } p_\varphi(x, \xi) = p(x, \xi + i\varphi'(x)) \ ,$$
$$\text{so that } \operatorname{Re} p_\varphi = q(\xi) - q(\varphi') + l.\xi + v + V \text{ and } \operatorname{Im} p_\varphi = (2{}^t\xi A + l)\varphi' \ .$$

The key to Carleman estimates is Hörmander's hypoellipticity condition :

$$p_\varphi = 0 \ \Rightarrow \ \{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} > 0 \ . \tag{11}$$

The operator P is elliptic near infinity, i.e. in some region $\left\{x \in \mathbb{R}^d \,|\, |x| > r\right\}$ for $r \gg R$. Without loss of generality (translating Ω and taking R larger) we may assume that Ω contains the following ball

$$\left\{x \in \mathbb{R}^d \,|\, |x| \leqslant R_1\right\} \subset \Omega \quad \text{ with } 0 < R_1 < R \ . \tag{12}$$

The less usual feature here is that (11) shall be be satisfied in the unbounded set

$$\mathcal{H} = \left\{x \in \mathbb{R}^d \,|\, |x| > R_0\right\} \quad \text{ with } 0 < R_0 < R_1 \ . \tag{13}$$

In the next subsection a global phase function is build in the usual form $\varphi = e^{\beta\psi}$ (cf. [Hör63, LR95, Bur98, Bur02a]). A global semiclassical Carleman estimate with this phase is proved in subsection 2.2 using a global Gårding inequality. Then a global semiclassical unique continuation estimate is deduced in subsection 2.3 using the properties of the phase.

2.1. **Phase for the Carleman weights.**

This section defines a global phase function $\varphi(x) = e^{\beta\psi(x)}$ where $\beta \geqslant 1$ shall be chosen large enough at the end of the section and ψ is the following radial smooth function on $\mathbb{R}^d$. For $R_2 > R$ to be given in the next lemma, $\psi(x) = R_2 - |x|$ for $|x| \geqslant R_1$, $\psi(x) = R_2$ for $|x| \leqslant R_0/2$, $\psi(x)$ decreases with $|x|$ for $R_0/2 \leqslant |x| \leqslant R_1$ and has a single inflexion point at $|x| = R_0$. In particular:

$$|\psi'| \geqslant 1 \text{ on } \mathcal{H} \quad \text{ and } \quad |\psi'(x)| = 1 \text{ for } |x| \geqslant R_1. \tag{14}$$

Since ψ is bounded from above and ψ', ψ'' are bounded, for any given β the functions φ, φ' and φ'' are bounded (and φ varies with $|x|$ from $e^{\beta R_2}$ to 0 without increasing).

We set $\lambda(x) = \beta e^{\beta\psi(x)}$ so that $\varphi' = \lambda\psi'$, $\lambda' = \beta\lambda\psi'$,

$$\operatorname{Re} p_\varphi = p - \lambda^2 q(\psi') \quad \text{ and } \quad \operatorname{Im} p_\varphi = \lambda \operatorname{Im} p_\psi \ .$$

Lemma 2.1. *$\exists R_2 > R$, $\forall\beta \geqslant 1$, $\forall x, \xi \in \mathbb{R}^d$,*

$$\operatorname{Re} p_\varphi(x,\xi) = 0 \quad \Rightarrow \quad \lambda(x) \geqslant 1,\ \xi = O(\lambda(x)) \text{ and } (1+|x|^2)^k = O(\lambda^2(x)) \ .$$

Proof. First note that $\operatorname{Re} p_\varphi(x,\xi) = 0$ implies

$$q_0|\xi|^2 \leqslant q(\xi) = \lambda^2 q(\psi') - l.\xi - v - V \leqslant M(\lambda^2 + |\xi| + 1)$$

with $M = \sup|q(\psi')| + \sup|l| + \sup|v| - \min\{0, \inf V\}$, which implies $\xi = O(1+\lambda)$.
Secondly $\lambda^2|\psi'|^2 \sup_x\|A\| \geqslant \lambda^2 q(\psi') \geqslant l.\xi + v + V \geqslant V - C(1+\lambda)$, so that

$$\begin{aligned}\operatorname{Re} p_\varphi(x,\xi) = 0,\ |x| \geqslant R_2 \ \Rightarrow\ & \lambda^2|\psi'|^2 \sup_x\|A\| \geqslant C_R(1+|x|^2)^k - C\left(1 + \tfrac{1+\lambda^2}{2}\right) \\ \Rightarrow\ & \lambda^2 \geqslant \frac{C_R(1+|x|^2)^k - 3C/2}{\sup_x\|A\| + C/2} = C_R'(1+|x|^2)^k - C' \ .\end{aligned}$$

We take $R_2 \geqslant R$ large enough for this last function to be greater than $C_R'(1+|x|^2)^k/2$ for all $|x| \geqslant R_2$ and for $C_R'(1+R_2^2)^k/2$ to be greater than 1. Thus $\operatorname{Re} p_\varphi(x,\xi) = 0$ and $|x| \geqslant R_2$ imply $\lambda(x) \geqslant 1$ and $(1+|x|^2)^k = O(\lambda^2)$. Moreover by mere definition of ψ, $|x| \leqslant R_2 \Rightarrow \psi(x) \geqslant 0 \Rightarrow \lambda(x) \geqslant 1$. Therefore the estimates in the lemma hold. $\square$

From now on, R_2 is fixed as in the previous lemma.

Lemma 2.2. *$\forall\beta \geqslant 1$, $p_\varphi(x,\xi) = 0 \ \Rightarrow\ \{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} = 4\beta\lambda^3\left(q(\psi')^2 + O(\beta^{-1})\right)$.*

Proof. The previous lemma, $\operatorname{Re} p_\varphi = 0 \Rightarrow V' = O(\lambda^2)$ and $\operatorname{Im} p_\varphi = 0 \Rightarrow \operatorname{Im} p_\psi = 0$ yield that $p_\varphi(x,\xi) = 0$ implies:

$$\begin{cases} \partial_x \operatorname{Re} p_\varphi = \partial_x p - \lambda^2\partial_x(q(\psi')) - 2\lambda\lambda' q(\psi') = -2\lambda\lambda' q(\psi') + O(\lambda^2) \\ \partial_x \operatorname{Im} p_\varphi = \lambda' \operatorname{Im} p_\psi + \lambda\partial_x \operatorname{Im} p_\psi = \lambda\partial_x \operatorname{Im} p_\psi = O(\lambda(1+|\xi|)) = O(\lambda^2) \ . \end{cases}$$

Plugging these equations in $\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} = 2({}^t\xi A + l)\partial_x \operatorname{Im} p_\varphi - 2\lambda {}^t\psi' A \partial_x \operatorname{Re} p_\varphi$ yields that $p_\varphi = 0$ implies $\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} = -2\lambda {}^t\psi' A[-2\lambda\lambda' q(\psi')] + O(\lambda^3)$. Plugging $\lambda' = \beta\lambda\psi'$ in this equation yields the lemma. $\square$

Now (14) yields $x \in \mathcal{H} \Rightarrow q(\psi')^2 + O(\beta^{-1}) \geqslant {q_0}^2/4$ for β large enough. Therefore, the lemmas yield that Hörmander's hypoellipticity condition holds uniformly on $\mathcal{H}$:

$$\exists\beta_0 \geqslant 1,\ \forall\beta \geqslant \beta_0,\ \forall x \in \mathcal{H}, \quad p_\varphi = 0 \ \Rightarrow\ \{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} \geqslant \beta q_0 > 0 \ . \tag{15}$$

From now on, we assume $\beta \geqslant \beta_0$.

2.2. **Global Carleman estimate.** For φ defined above, this section proves

Proposition 2.3. *$\forall\mathcal{U} \Subset \mathcal{H}$, $\exists c_0 > 0$, $\exists h_0 > 0$, $\forall h \in (0, h_0]$, $\forall u \in \mathcal{C}_0^\infty(\mathcal{U})$,*

$$\int_{\mathbb{R}^d} |P_\varphi u|^2 \geqslant c_0 h \int_{\mathbb{R}^d} |h\nabla u|^2 + (1+|x|^2)^k |u|^2 \ .$$

Note that the integral on the right hand side can be written as $\|M^{1/2}u\|^2$ (squared L^2 norm) with the notations

$$m(x,\xi) = |\xi|^2 + (1+|x|^2)^k \text{ and } M = m(x, hD) = M^* \geqslant I \ .$$

The proof of proposition 2.3 is based on the following Gårding's elliptic inequality (simpler than the sharp version for $m = 1$ in [DS99, theorem 7.12] e.g.).

Proposition 2.4. *If the symbol $a \in S(m)$ (with weight m) satisfies the lower bound $\exists\alpha > 0$, $\forall(x,\xi) \in \mathcal{H} \times \mathbb{R}^d$, $a(x,\xi) \geqslant \alpha m(x,\xi)$, then any semiclassical operator $A_h = a(x, hD) + hr_a(x, hD)$ (with principal symbol a and remainder $r_a \in S(m)$) satisfies the lower bound: $\forall\mathcal{U} \Subset \mathcal{H}$, $\forall\varepsilon > 0$, $\exists h_0 > 0$, $\forall h \in (0, h_0]$, $\forall u \in \mathcal{C}_0^\infty(\mathcal{U})$,*

$$\operatorname{Re} \int_{\mathbb{R}^d} \overline{u} A_h u \geqslant (\alpha - \varepsilon)\|M^{1/2}u\|^2$$

Proof. Choose $\chi \in S(1)$ on $\mathbb{R}^d$ such that $\chi = 1$ on $\overline{\mathcal{U}}$ and $\operatorname{supp}\chi \subset \mathcal{H}$ (e.g. $\chi \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that $\chi = 0$ in a compact neighborhood of $\mathbb{R}^d \setminus \mathcal{H}$ contained in $\mathbb{R}^d \setminus \overline{\mathcal{U}}$ and $\chi = 1$ outside a larger such neighborhood). Set $\beta = \alpha - \varepsilon/2$ and $b = \chi\sqrt{a-\beta m} \in S(m^{1/2})$ (well defined since $a - \beta m \geqslant (\varepsilon/2)m \geqslant \varepsilon/2 > 0$ on $\operatorname{supp}\chi \times \mathbb{R}^d$). The symbolic calculus in [DS99, Chapter 7] yields $\chi\#a\#\chi = \beta\chi\#m\#\chi + b\#b + hr_b$ with $r_b \in S(m)$. For any $u \in \mathcal{C}_0^\infty(\mathcal{U})$, we compute the L^2 hermitian product $\langle A_h u, u\rangle = \beta\|M^{1/2}(\chi u)\|^2 + \|b(x,hD)\|^2 + h\langle r(x,hD)u,u\rangle$ with $r = r_a + r_b \in S(m)$.

We may write $\langle r(x,hD)u,u\rangle = \langle R_0u_0,u_0\rangle$ with $R_0 = M^{-1/2}r(x,hD)M^{-1/2}$ and $u_0 = M^{1/2}u$. Since $M^{-1/2}$ is a pseudo-differential operator with symbol in $S(m^{-1/2})$ (cf. theorem A.1 in the appendix), R_0 is a pseudo-differential operator with symbol in $S(1)$, so that (cf. [DS99, Thm 7.11]) R_0 is bounded on L^2 by some constant $\rho > 0$. Therefore $|\langle r(x,hD)u,u\rangle| \leqslant \rho\|M^{1/2}u\|^2$.

Hence $\operatorname{Re}\langle A_hu,u\rangle \geqslant \beta\|M^{1/2}(\chi u)\|^2 - h\rho\|M^{1/2}u\|^2$. Setting $h_0 = \frac{\varepsilon}{2\rho}$ completes the proof: for all $h \leqslant h_0$, $\operatorname{Re}\langle A_hu,u\rangle \geqslant (\beta - h_0\rho)\|M^{1/2}u\|^2 = (\alpha-\varepsilon)\|M^{1/2}u\|^2$. $\square$

To apply Gårding's inequality, we shall need the following lower bound.

Lemma 2.5. *$\exists\gamma > 0$, $\exists\alpha > 0$, $\forall x \in \mathcal{H}$, $\forall\xi \in \mathbb{R}^d$,*

$$\gamma\left(\frac{|\operatorname{Re}p_\varphi|^2}{m} + |\operatorname{Im}p_\varphi|^2\right) + \{\operatorname{Re}p_\varphi, \operatorname{Im}p_\varphi\} \geqslant \alpha m\ .$$

Proof. Recall $\operatorname{Re}p_\varphi - V = q(\xi) - q(\varphi') + l.\xi + v \geqslant q_0|\xi|^2 - q_1(1+|\xi|)$ where $q_2 = \sup(|l| + |v| + |A|)$ is bounded by assumption on p and $q_1 = q_2(1+\sup|\varphi'|^2)$ is bounded (for β given) by definition of φ. For all $\rho > 0$ and $\kappa \in (0,q_0)$, there exists δ large enough so that the previous quadratic polynomial in the variable $|\xi|$ is bounded from below by $\kappa|\xi|^2 + \rho$ for $|\xi| \geqslant \delta$. Setting $\rho = \sup_{|x|\leqslant R} V + C_R(1+|R|^2)^k$, (1) implies $V + \rho \geqslant C_R(1+|x|^2)^k$ for all $x \in \mathbb{R}^d$. The corresponding δ satisfy:

$$\forall x \in \mathbb{R}^d,\ \forall\xi \in \mathbb{R}^d, \quad |\xi| \geqslant \delta \Rightarrow \operatorname{Re}p_\varphi \geqslant \kappa|\xi|^2 + C_R(1+|x|^2)^k\ .$$

Similarly, $|x| \geqslant R$ and $|\xi| \leqslant \delta$ imply $\operatorname{Re}p_\varphi \geqslant q_0|\xi|^2 - q_1(1+\delta) + C_R(1+|x|^2)^k$. For all $\rho' > 0$ and $\kappa' \in (0, C_R)$, there exists δ' large enough so that $C_R(1+|x|^2)^k - \rho' > \kappa'(1+|x|^2)^k$ for $|x| \geqslant \delta'$. Taking $\delta' > R$ corresponding to $\rho' = (1+\delta)q_1$ yields :

$$\forall x \in \mathbb{R}^d,\ \forall\xi \in \mathbb{R}^d, \quad |x| \geqslant \delta' \text{ and } |\xi| \leqslant \delta \Rightarrow \operatorname{Re}p_\varphi \geqslant q_0|\xi|^2 + \kappa'(1+|x|^2)^k\ .$$

Therefore in both cases, $|\xi| \geqslant \delta$ or $|x| \geqslant \delta'$, the symbol $\operatorname{Re}p_\varphi$ is bounded from below by a positive multiple of m. Since $\{\operatorname{Re}p_\varphi, \operatorname{Im}p_\varphi\} \in S(m)$, there exists $\gamma > 0$ such that: $\forall x \in \mathbb{R}^d$, $\forall\xi \in \mathbb{R}^d$,

$$|x| \geqslant \delta' \text{ or } |\xi| \geqslant \delta \Rightarrow \gamma\left(\frac{|\operatorname{Re}p_\varphi|^2}{m} + |\operatorname{Im}p_\varphi|^2\right) + \{\operatorname{Re}p_\varphi, \operatorname{Im}p_\varphi\} \geqslant m\ .$$

Now there is an open neighborhood $\mathcal{N}$ of $\{p_\varphi = 0\} \cap \mathcal{H}$ such that $\{\operatorname{Re}p_\varphi, \operatorname{Im}p_\varphi\} \geqslant \kappa q_0/2$ on $\mathcal{N}$, according to the uniform hypoellipticity condition (15). Since the complementary set $K = \{|\xi| \leqslant \delta, |x| \leqslant \delta'\} \cap \mathcal{H}$ is compact, $\alpha_1 = \min_{x\in K}\kappa q_0/(2m)$ and $\alpha_2 = \min_{x\in K\setminus\mathcal{N}}\gamma|p_\varphi|^2/m^2$ are positive. Setting $\alpha = \min\{1,\alpha_1,\alpha_2\}$ completes the proof of the lemma. $\square$

Proof of proposition 2.3. Let $A = (P_\varphi + P_\varphi^*)/2$ and $B = (P_\varphi - P_\varphi^*)/(2i)$ so that $P_\varphi = A + iB$, $A = A^*$ and $B = B^*$. Setting $C = \frac{i}{h}[A,B]$ yields

$$\|P_\varphi u\|^2 = \|Au\|^2 + \|Bu\|^2 + h\langle Cu,u\rangle = \operatorname{Re}\langle (A^2+B^2+hC)u,u\rangle\ .$$

The Gårding inequality in proposition 2.4 applies to $E = \gamma\left((M^{-1/2}A)^2 + B^2\right) + C$, thanks to the estimate of its principal symbol in lemma 2.5. With $c_0 = \alpha - \varepsilon$, it is

$$\mathrm{Re}\langle Eu, u\rangle = \gamma\left(\|M^{-1/2}Au\|^2 + \|Bu\|^2\right) + \mathrm{Re}\langle Cu, u\rangle \geqslant c_0\|M^{1/2}u\|^2 \ .$$

Now $\|M^{-1/2}\| \leqslant 1$, so that $\mathrm{Re}\langle Cu, u\rangle \geqslant c_0\|M^{1/2}u\|^2 - \gamma\left(\|Au\|^2 + \|Bu\|^2\right)$. Hence

$$\|P_\varphi u\|^2 \geqslant (1-\gamma h)\left(\|Au\|^2 + \|Bu\|^2\right) + c_0 h\|M^{1/2}u\|^2 \ .$$

Setting $h_0 = 1/\gamma$ completes the proof of the proposition. □

2.3. **Unique continuation estimate.** This section proves the following generalized version of theorem 1.2.

Theorem 2.6. *For any non empty open $\Omega \subset \mathbb{R}^d$, $\exists C_0 > 0$, $\exists h_0 > 0$, $\forall h \in (0, h_0)$,*

$$\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^d), \int_{\mathbb{R}^d} |h\nabla u|^2 + (1+|x|^2)^k|u|^2 \leqslant e^{C_0/h}\left(\int_\Omega |u|^2 + \int_{\mathbb{R}^d}|Pu|^2\right).$$

The main step in the proof is the following proposition where $\varphi = e^{\beta\psi}$ was defined in section 2.1.

Proposition 2.7. $\forall\beta > \beta_0$, $\exists C_1 > 0$, $\exists h_0 > 0$, $\forall h \in (0, h_0]$, $\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$h\int_{\mathbb{R}^d} U \leqslant C_1 e^{2(\varphi(R_0)-\varphi(R_1))/h}\left(h\int_{|x|\leqslant R_1} U + \int_{\mathbb{R}^d}|Pu|^2\right) \ ,$$

with the abbreviation $U(x) = |h\nabla u(x)|^2 + (1+|x|^2)^k|u(x)|^2$ and $\varphi(|x|) = \varphi(x)$.

Proof. Choose $\chi \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that $\chi = 1$ for $|x| \geqslant R_1$ and $\operatorname{supp}\chi \subset \mathcal{H}$. There is a $\mathcal{U} \Subset \mathcal{H}$ such that $\operatorname{supp}\chi \subset \mathcal{U}$. According to prop. 2.3, $\exists c_0 > 0$, $\exists h_0 > 0$, $\forall h \in (0, h_0]$, $\forall f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |P_\varphi(\chi f)|^2 \geqslant c_0 h\|M^{1/2}(\chi f)\|^2 \ . \tag{16}$$

For any $u \in \mathcal{C}_0^\infty(\mathcal{U})$, we consider $f = e^{\varphi/h}u$. Note that $h\nabla u = (h\nabla f - f\nabla\varphi)e^{-\varphi/h}$ implies $e^{2\varphi/h}|h\nabla u|^2 \leqslant 2(|h\nabla f|^2 + |f|^2)(1+\sup|\nabla\varphi|^2)$ so that

$$\begin{aligned} e^{2\varphi(R_1)/h}\int_{|x|\geqslant R_1} U &\leqslant (3 + 2\sup|\nabla\varphi|^2)\int_{|x|\geqslant R_1} |h\nabla f|^2 + (1+|x|^2)^k|f|^2 \\ &\leqslant (3 + 2\sup|\nabla\varphi|^2)\|M^{1/2}(\chi f)\|^2 \ . \end{aligned}$$

Using (16) and $P_\varphi(\chi f) = e^{\varphi/h}P(\chi u)$, we deduce, setting $c_1 = c_0/(3 + 2\sup|\nabla\varphi|^2)$

$$\int_{\mathbb{R}^d} |P(\chi u)|^2 e^{2\varphi/h} \geqslant c_1 h e^{2\varphi(R_1)/h}\int_{|x|\geqslant R_1} U \ . \tag{17}$$

Now $\chi = 0$ for $|x| \leqslant R_0$ and $\chi = 1$ for $|x| \geqslant R_1$ imply

$$\begin{cases} \int |\chi Pu|^2 e^{2\varphi/h} \leqslant e^{2\varphi(R_0)/h}\sup|\chi|^2 \int |Pu|^2 \\ \int |[\chi, P]u|^2 e^{2\varphi/h} \leqslant h e^{2\varphi(R_0)/h}\int_{R_0\leqslant|x|\leqslant R_1} U \end{cases} ,$$

so that, for some $c_2 > 0$ which depends only on h_0, φ and χ,

$$\int_{\mathbb{R}^d} |P(\chi u)|^2 e^{2\varphi/h} \leqslant c_2 e^{2\varphi(R_0)/h}\left(h\int_{|x|\leqslant R_1} U + \int |Pu|^2\right)$$

Plugging this in (17) completes the proof the proposition with $C_1 = h_0 + c_2/c_1$. □

Proof of theorem 2.6. Since $p(x,\xi) \geqslant q_0|\xi|^2/2$ for $|\xi|$ large enough, P is elliptic. Therefore (12) implies for h_0 small enough: $\exists C_2 > 0$, $\forall h \in (0, h_0]$, $\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$\int_{|x|\leqslant R_1} U \leqslant C_2 \left(\int_\Omega |u|^2 + \int_\Omega |Pu|^2 \right) .$$

Plugging this in prop. 2.7, taking $C_0 > 2(\varphi(R_0) - \varphi(R_1)) > 0$ and h_0 small enough completes the proof. □

3. Sums of eigenfunctions

In this section M denotes either $\mathbb{R}^d$ or a compact manifold as in theorem 1.3. Thus (2) writes exactly as (3). We introduce a time variable t, space-time manifolds $M_t = (-t, t) \times M$ for any $t > 0$, $M_\infty = \mathbb{R}_t \times M$ and $M_0 = \{t = 0\} \times M$.

When $M = \mathbb{R}^d$, the operator P is the same as in the previous section. When M is a compact manifold, P denotes an operator with Dirichlet boundary condition which is locally the same as in the previous section.

3.1. **Reduction to a unique continuation estimate with an extra elliptic variable.**

In this section, we assume that P is self-adjoint. The main results on sums of eigenfunctions of P reduces to the following estimate on the space time semiclassical Schrödinger operator $-h^2\partial_t^2 + P$ (where t is an extra elliptic variable).

Proposition 3.1. *Theorems 1.1 (where $M = \mathbb{R}^d$) and 1.3 (where M is compact manifold) follow from:*
for any non empty open subset Ω of M_0, $\exists T_1 > 0$, $\exists T_2 > 0$, $\forall \gamma > 0$, $\exists C > 0$, $\exists h_0 > 0$, $\forall h \in (0, h_0]$, $\forall f \in \mathcal{C}_0^\infty(M_\infty)$ such that $f(0,\cdot) = 0$,

$$\int_{M_{T_1}} F \leqslant Ce^{C/h} \left(\int_\Omega |\dot{f}_0|^2 + \int_{M_{T_2}} |(-h^2\partial_t^2 + P)f|^2 \right) + Ce^{-\gamma/h} \int_{M_{T_2}} F , \tag{18}$$

with the abbreviations $\dot{f}_0(x) = h\partial_t f(0, x)$ and $F = |h\partial_t f|^2 + |h\nabla_x f|^2 + (1+|x|^2)^k|f|^2$ (n.b. when M is a compact manifold the weight $(1 + |x|^2)^k$ is withdrawn from F).

Proof. Since p and P are bounded from below, adding a constant without loss of generality for theorems 1.1 and 1.3, we assume that their lower bound is positive.

Let $E > 0$, $\phi \in \mathcal{C}_0^\infty(M)$ and $u = \mathbf{1}_{P<E}\,\phi$. Since $\mathcal{C}_0^\infty(M)$ is dense in $L^2(M)$, it is enough to prove (3) for such u.

Let dE_λ denote the projection valued measure associated to the positive self-adjoint operator $\sqrt{P}$. The *cardinal hyperbolic sine function* is shc $\in \mathcal{C}^\infty(\mathbb{R})$ defined by: $\mathrm{shc}(0) = 1$ and $\mathrm{shc}(t) = (e^t - e^{-t})/(2t)$ for $t \neq 0$.

Let $F_E(t,\lambda) = t\,\mathrm{shc}(t\lambda)\,\mathbf{1}_{\lambda<\sqrt{E}}$. For all j and k in $\mathbb{N}$, $\lambda^j\partial_t^k F_E \in \mathcal{C}(\mathbb{R}_t; L^\infty(\mathbb{R}_\lambda))$. Therefore the function f defined by:

$$f(t,x) = F_E(t/h, \sqrt{P})\phi = \sum_{k\in\mathbb{N}} \frac{(t/h)^{2k+1}}{(2k+1)!} P^k u ,$$

satisfies $f \in H^j(M_T)$ for all $j \in \mathbb{N}$ and $T > 0$. Moreover, since $\phi \in \mathrm{D}((\sqrt{P})^j)$ for all $j \in \mathbb{N}$, we have $f(t,\cdot) \in \cap_{j\in\mathbb{N}} \mathrm{D}((\sqrt{P})^j) \subset H_0^1(M)$ for all t.

Since $F_E(0,\lambda) = 0 \Rightarrow f(0,\cdot) = 0$ and $f(t,\cdot) \in H_0^1(M) \Rightarrow f(t,x) = 0$ for $x \in \partial M$, we may approximate f by functions in $\mathcal{C}_0^\infty(M_\infty)$ such that $f(0,\cdot) = 0$ so that the hypothesis of proposition 3.1 still applies to this f. Due to the ellipticity of $-h^2\partial_t^2 + P$, we have $\int_{M_{T_2}} F \leqslant c_\varepsilon \int_{M_{T_2+\varepsilon}} |(-h^2\partial_t^2 + P)f|^2 + |f|^2$ for any $\varepsilon > 0$. Therefore (18) still holds with F replaced by $|f|^2$. The resulting equation writes, since $\partial_t^2 F_E = \lambda^2 F_E \Rightarrow h^2\partial_t^2 f = Pf$ and $(t\,\mathrm{shc}(t))'_{\rceil t=0} = \cosh(0) = 1 \Rightarrow \dot f_0 = u$:

$$\int_{M_{T_1}} |f|^2 \leqslant Ce^{C/h}\int_\Omega |u|^2 + Ce^{-\gamma/h}\int_{M_{T_2}} |f|^2 \ . \tag{19}$$

Now $\mathrm{shc} \geqslant 1$, $t\,\mathrm{shc}(t\lambda) \leqslant e^{t(1+\lambda)}/2$ for $t \geqslant 0$ and $\lambda \geqslant 0$, and

$$\int_{M_T} |f|^2 = \int_{-T}^{T}\!\!\int |F_E(\frac{t}{h},\lambda)|^2 d(E_\lambda\phi,\phi)\,dt = 2h\int_0^{\frac{T}{h}}\int_0^{\sqrt{E}} |t\,\mathrm{shc}(t\lambda)|^2 d(E_\lambda\phi,\phi)\,dt \ ,$$

imply $\frac{2}{h^2}\frac{T^3}{3}\int_M |u|^2 \leqslant \int_{M_T}|f|^2 \leqslant Te^{2T(1+\sqrt{E})/h}\int_M |u|^2$. Therefore (19) implies

$$\left(\frac{2}{3}T_1^3 - T_2Ch^2 e^{(2T_2(1+\sqrt{E})-\gamma)/h}\right)\int_M |u|^2 \leqslant Ch^2 e^{C/h}\int_\Omega |u|^2 \ .$$

Taking $\gamma > 2T_2(1+\sqrt{E})$, C large enough and h_0 small enough yields (3). □

3.2. **First case: compact potential well.** In this section we prove theorem 1.1. Indeed, we shall prove that the hypothesis of proposition 3.1 still holds for all $T_2 > T_1 > 0$. More precisely, we shall prove the estimate (18) by combining two estimates with the same structure stated in the following lemmas.

Let $T > T_2$. We introduce a space time semiclassical Schrödinger operator $Q = -h^2\partial_t^2 + W(t) + P$ where $W \in \mathcal{C}^\infty(\mathbb{R})$ satisfies $W(t) = 0$ for $|t| \leqslant T_2$ and $W(t) = (1+|t|^2)^k$ for $|t| > T$ large enough. Note that Q satisfies (1) on $M_\infty = \mathbb{R}^{d+1}$. We still denote by $\varphi = e^{\beta\psi}$ the corresponding phase on M_∞ defined in section 2.1 with $0 < R_0 < R_1 < T_1$. A slight modification of the proof of proposition 2.7 yields: $\forall\beta > \beta_0$, $\exists C_0 > 0$, $\exists h_0 > 0$, $\forall h \in (0, h_0]$, $\forall f \in \mathcal{C}_0^\infty(M_\infty)$,

$$hC_0\int_{M_{T_1}} F \leqslant e^{2D(\beta)/h}\left(\int_{B_{R_1}} F + \int_{M_{T_2}} |Qf|^2\right) + e^{-2\delta(\beta)/h}\int_{M_{T_2}\setminus M_{T_1}} F \ , \tag{20}$$

where $B_{R_1} = \{y \in M_\infty \,|\, |y| \leqslant R_1\}$, $D(\beta) = \varphi(R_0) - \varphi(R_1) = e^{\beta\psi(R_0)} - e^{\beta\psi(R_1)}$ and $\delta(\beta) = \varphi(R_1) - \varphi(T_1) = e^{\beta\psi(R_1)} - e^{\beta\psi(T_1)}$ are positive functions increasing to infinity. Indeed, it is enough to modify χ in the proof as follows. As before, choose $\chi_1 \in \mathcal{C}^\infty(M_\infty)$ such that $\chi_1 = 1$ out of B_{R_1} and $\mathrm{supp}\,\chi_1 \subset \mathcal{H}$. Now choose $\chi_2 \in \mathcal{C}^\infty(\mathbb{R})$ such that $\chi_2 = 1$ on $[-T_1, T_1]$ and $\mathrm{supp}\,\chi_2 \subset (-T_2, T_2)$. Defining $\chi(t,x) = \chi_1(t,x)\chi_2(t)$, the proof is easily completed.

For any $\gamma > 0$, take β such that $2\delta(\beta) \geqslant \gamma$, then take $C \geqslant \max\{C_0, 2D(\beta)\}$. Thus (20) proves this lemma where $Q = -h^2\partial_t^2 + P$ since $W = 0$ on M_{T_2}:

Lemma 3.2. *$\forall\gamma > 0$, $\exists C > 0$, $\exists h_0 > 0$, $\forall h \in (0, h_0]$, $\forall f \in \mathcal{C}_0^\infty(M_\infty)$,*

$$\int_{M_{T_1}} F \leqslant Ce^{C/h}\left(\int_{B_{R_1}} |f|^2 + \int_{M_{T_2}} |Qf|^2\right) + Ce^{-\gamma/h}\int_{M_{T_2}} F \ . \tag{21}$$

Let $0 < T_0 < \tilde R_1 < \tilde R_2 < \tilde R_3$ and consider the closed balls $\tilde B_i$ with center $y_0 = (t_0, x_0) = (-T_0, 0)$ and radii $\tilde R_i$, $i \in \{1,2,3\}$. Let $\tilde B_1^+ = \tilde B_1 \cap \{t > 0\}$ and $\tilde B_3^0 = \tilde B_3 \cap \{t = 0\}$. The semiclassical Carleman estimate with boundary of [LR95] proves this lemma with the abbreviations of proposition 3.1:

Lemma 3.3. *$\forall\tilde\gamma > 0$, $\exists\tilde C > 0$, $\exists h_0 > 0$, $\forall h \in (0, h_0]$, $\forall f \in \mathcal{C}_0^\infty(M_\infty)$, such that $f(0,\cdot) = 0$,*

$$\int_{\tilde B_1^+} F \leqslant \tilde Ce^{\tilde C/h}\left(\int_{\tilde B_3^0} |\dot f_0|^2 + \int_{\tilde B_3} |Qf|^2\right) + \tilde Ce^{-\tilde\gamma/h}\int_{\tilde B_3\setminus\tilde B_1} F \ . \tag{22}$$

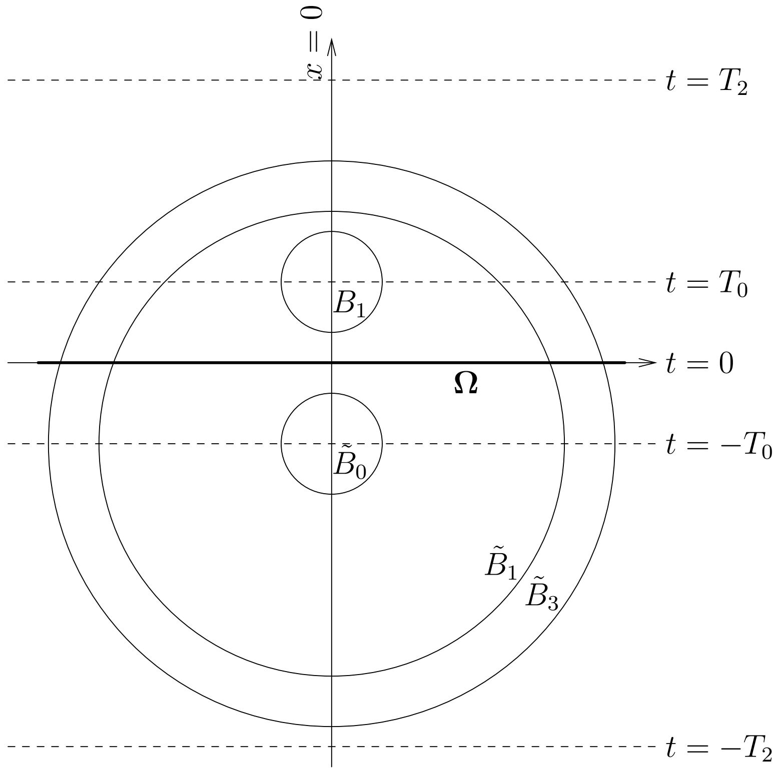

FIGURE 1

Proof. Let $K = \tilde{B}_3 \cap \{t \geqslant 0\}$. Define the phase $\tilde{\varphi}(y) = e^{\beta(\tilde{R}_3 - |y-y_0|)}$ for $\beta > \beta_0 > 0$. It satisfies $\partial_t \tilde{\varphi} \neq 0$ on K. As in [LR95], for β_0 large enough this phase satisfies Hörmander's hypoellipticity condition (11) for Q on $K \times \mathbb{R}^{d+1}$. Thus proposition 3.1 in [LR95] proves:
$\forall \beta > \beta_0$, $\exists c_0 > 0$, $\exists h_0 > 0$, $\forall h \in (0, h_0]$, $\forall f \in \mathcal{C}_0^\infty(\tilde{B}_3)$, such that $f(0, \cdot) = 0$,

$$hc_0 \int_K F e^{2\tilde{\varphi}/h} \leqslant h \int_{\tilde{B}_3^0} |\dot{f}_0|^2 e^{2\tilde{\varphi}/h} + \int_K |Qf|^2 e^{2\tilde{\varphi}/h} \ . \tag{23}$$

Choose $\chi \in \mathcal{C}_0^\infty(\tilde{B}_3)$ such that $\chi = 1$ on $\tilde{B}_2 \supset \tilde{B}_1^+$. Replacing f in (23) by χf for any $f \in \mathcal{C}_0^\infty(M_\infty)$ such that $f(0, \cdot) = 0$ yields as in proposition 2.7

$$hC_0 e^{\frac{2}{h} e^{\beta(\tilde{R}_3 - \tilde{R}_1)}} \int_{\tilde{B}_1^+} F \leqslant e^{\frac{2}{h} e^{\beta \tilde{R}_3}} \left(\int_{\tilde{B}_3^0} |\dot{f}_0|^2 + \int_{\tilde{B}_3} |Qf|^2 \right) + e^{\frac{2}{h} e^{\beta(\tilde{R}_3 - \tilde{R}_2)}} \int_{\tilde{B}_3 \setminus \tilde{B}_2} F \ .$$

Now the proof can be completed as for lemma 3.2 by taking β large enough. □

In this last step, called "semiclassical propagation of smallness" in the introduction, we combine the two lemmas to prove (18).

We first adjust the geometry. Translating f if needed, lemma 3.2 still holds with B_{R_1} replaced by the ball B_1 with center $(T_0, 0)$ and radius $R_1 < T_0$. We ensure that (cf. figure 1)

$$\tilde{B}_3^0 \subset \Omega, \quad B_1 \subset \tilde{B}_1^+, \quad \tilde{B}_3 \subset M_{T_2} \ .$$

The first inclusion is obtained by translating f along M and taking $\tilde{R}_3$ small enough, the second by taking $2T_0 + \tilde{R}_1 > R_1$ and the last by taking $T_2 > \tilde{R}_3 + T_0$.

With these inclusions, (22) implies

$$\int_{B_1} F \leqslant \tilde{C}e^{\tilde{C}/h}\left(\int_\Omega |\dot{f}_0|^2 + \int_{M_{T_2}} |Qf|^2\right) + \tilde{C}e^{-\tilde{\gamma}/h}\int_{M_{T_2}} F\ .$$

Plugging this in (21) with B_{R_1} replaced by B_1 yields

$$\int_{M_{T_1}} F \leqslant Ce^{C/h}\left(1+\tilde{C}e^{\tilde{C}/h}\right)\left(\int_\Omega |\dot{f}_0|^2 + \int_{M_{T_2}} |Qf|^2\right) + C\left(\tilde{C}e^{-(\tilde{\gamma}-C)/h} + e^{-\gamma/h}\right)\int_{M_{T_2}} F\ .$$

To complete the proof of (18) we note that the last parenthesis can be made an arbitrary exponentially small factor by choosing γ first, then $\tilde{\gamma}$ depending on C.

3.3. **Second case: compact manifold.** In this section we prove theorem 1.3. As in the previous section, we shall prove that the hypothesis of proposition 3.1 still holds for all $T_2 > T_1 > 0$. The "semiclassical propagation of smallness" introduced above allows to prove the global estimate (18) by combining local estimates with the same structure.

The needed local estimates are all obtained from [LR95, sect. 3 (40)] by choosing β large enough as in the proof lemma 3.2 above so that γ is arbitrarily large. The geometry is the same as in [LR95, sect. 3]. The propagation of smallness from Ω is initiated by a time boundary estimate as in lemma 3.3 and ends with spatial boundary estimates (cf. [LR95, last page]). We now concentrate our explanation on the interior propagation using the geometrical setting of [LR95, p. 354].

We recall there are balls B_n, $n = 0, \dots, N$, with a common radius r such that B_{n+1} is included in the ball $3B_n$ with same center as B_n but radius $3r$. A semiclassical Carleman estimate with "radial" phase ($-\psi$ is the distance from the center of B_n, cf. [LR95]) yields, as in lemma 3.2, with $F = |h\partial_t f|^2 + |h\nabla_x f|^2 + |f|^2$,
$\forall\gamma_n > 0$, $\exists C_n > 0$, $\exists h_0 > 0$, $\forall h \in (0, h_0]$, $\forall f \in \mathcal{C}_0^\infty(M_\infty)$,

$$\int_{B_{n+1}} F \leqslant \int_{3B_n} F \leqslant C_n e^{C_n/h}\left(\int_{B_n} F + \int_{M_{T_2}} |Qf|^2\right) + C_n e^{-\gamma_n/h}\int_{M_{T_2}} F\ .$$

Setting $\alpha_n = \int_{B_n} |F|^2$, $A = \int_{M_{T_2}} |Qf|^2$ and $B = \int_{M_{T_2}} F$, we have to deduce $\forall\gamma > 0$, $\exists C > 0$, $\exists h_0 > 0$, $\forall h \in (0, h_0]$, $\forall f \in \mathcal{C}_0^\infty(M_\infty)$, $\alpha_N \leqslant Ce^{C/h}(\alpha_0 + A) + Ce^{-\gamma/h}B$. N.b. this deduction is a substitute for [LR95, lemma 3.4]. Reasoning by recurrence, it is enough to prove it for $N = 2$. The hypothesis writes

$$\alpha_1 \leqslant C_0 e^{C_0/h}(\alpha_0 + A) + C_0 e^{-\gamma_0/h}B \text{ and } \alpha_2 \leqslant C_1 e^{C_1/h}(\alpha_1 + A) + C_1 e^{-\gamma_1/h}B\ .$$

It implies $\alpha_2 \leqslant C_0 C_1 e^{(C_0+C_1)/h}(\alpha_0 + A) + \left(C_0 C_1 e^{-(\gamma_0 - C_1)/h} + C_1 e^{-\gamma_1/h}\right)B$. The conclusion is obtained by choosing $\gamma_1 > \gamma$ first, then $\gamma_0 > \gamma + C_1$ and finally $C > (C_0 + 1)(C_0 + C_1)$.

4. Homogeneous potentials: sharpness and non-controllability

4.1. **Semiclassical reduction by scaling.** In this section, we consider a smooth potential well V which satisfies (1) and is positively homogeneous of degree $2k$, i.e.

$$\forall\varepsilon > 0, \quad V(\varepsilon x) = \varepsilon^{2k}V(x)\ , \tag{24}$$

and prove that corollaries 1.6 and 1.7 are still valid for the operator $A = -\Delta + V$ (cf. remark 1.8). They reduce to theorems 1.1 and 2.6 which concern the semiclassical operator $P = -h^2\Delta + V$ by some change of variable $u(x) = v(x/\varepsilon)$ with $\varepsilon > 0$. Since $(Pu - \lambda u)(x) = \big(-\big(\frac{h}{\varepsilon}\big)^2\Delta v + \varepsilon^{2k}Vv - \lambda v\big)(x/\varepsilon)$, the scale ε must satisfy

$$\left(\frac{h}{\varepsilon}\right)^2 = \varepsilon^{2k} = \frac{1}{\mu^2} \ . \tag{25}$$

Indeed, setting $1/h = \mu^{1+1/k}$ and defining the operator D_ε of dilation with scale $\varepsilon = \mu^{-1/k}$ by $(D_\varepsilon u)(x) = u(\varepsilon x)$ yields

$$D_\varepsilon(P-\lambda)u = \mu^{-2}(A - \mu^2\lambda)v, \quad \text{ for } v(x) = u(\varepsilon x) \ .$$

In particular $u \in \mathbf{1}_{P<E}\, L^2(\mathbb{R}^d)$ is equivalent to $v \in \mathbf{1}_{A<\mu^2 E}\, L^2(\mathbb{R}^d)$.

To prove corollary 1.6, we take $E_0 = 1$, $V(x) = |x|^{2k}$ and $\Omega \subset \Gamma$ in theorem 1.1. We may assume $h_0 < 1$ and let $h_0 = \varepsilon_0^{k+1} = \mu_0^{-(1+1/k)}$. Performing the change of variable in the estimate (2), and dividing by ε^d, yields

$$\forall \mu \geqslant \mu_0, \forall v \in \mathbf{1}_{A<\mu^2}\, L^2(\mathbb{R}^d), \int_{\mathbb{R}^d} |v(x)|^2 dx \leqslant Ce^{C\mu^{1+1/k}} \int_{\varepsilon^{-1}\Omega} |v(x)|^2 dx \ .$$

Since $\varepsilon_0 < 1$, $\varepsilon x \in \Omega$ implies $x \in \Gamma$, so that the last integral is bounded by the same integral over Γ. Increasing C, this proves (6) for all $\mu \geqslant \mu_0$. Increasing again C to $C\exp(C\mu_0^{1+1/k})$ completes the proof of corollary 1.6.

To prove corollary 1.7, we take $P = -h^2\Delta + |x|^{2k} - 1$ in theorem 2.6. Scaling as before so that $D_\varepsilon Pu = \mu^{-2}(A - \mu^2)v$ and using (25) yields $\forall \mu \geqslant \mu_0$, $\forall v \in \mathrm{D}(A)$,

$$\int_{\mathbb{R}^d} \frac{1}{\mu^2}|\nabla v|^2 + (1 + \frac{1}{\mu^2}|x|^{2k})|v|^2 \leqslant Ce^{C\mu^{1+1/k}} \left(\int_\Gamma |v|^2 + \frac{1}{\mu^4}\int_{\mathbb{R}^d} |(A-\mu^2)v|^2 dx \right) \ .$$

Increasing C, this proves (7) for $\lambda \geqslant \mu_0^2$. For $\lambda \leqslant -\mu_0^2 < 0$, (7) without $\int_\Gamma |v|^2$ is a standard elliptic estimate. Since $\|(A-\lambda)v\|^2 = \|(A - \operatorname{Re}\lambda)v\|^2 + \|(\operatorname{Im}\lambda)v\|^2 \geqslant \|(A - \operatorname{Re}\lambda)v\|^2$, we are left with proving (7) for $\lambda \in \mathbb{R}$, $|\lambda| < \mu_0^2$. Arguing by contradiction, we consider sequences (v_n) in $\mathrm{D}(A)$ and (λ_n) in $(-\mu_0^2, \mu_0^2)$ such that

$$\int_\Gamma |v_n|^2 + \int_{\mathbb{R}^d} |(A-\lambda_n)v_n|^2 = o(1) \text{ and } \int_{\mathbb{R}^d} |\nabla v_n|^2 + (1 + |x|^{2k})|v_n|^2 = 1 \ .$$

Since $[-\mu_0^2, \mu_0^2]$ is compact and $\big\{v \in L^2(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |\nabla v|^2 + (1+|x|^{2k})|v|^2 \leqslant 1\big\}$ is compact for the L^2 topology, passing to subsequences if needed, we may assume that $\lambda_n \to \lambda$ and $\int_{\mathbb{R}^d} |v_n - v|^2 \to 0$. Now $\int_\Gamma |v|^2 = \lim \int_\Gamma |v_n|^2 = 0$ and $(A-\lambda)v = \lim(A - \lambda_n)v_n = 0$ in the distributions topology. Hence $v = 0$ by unique continuation for second order elliptic operators. Integrating by parts yields the contradiction

$$1 = \int_{\mathbb{R}^d} |\nabla v_n|^2 + (1+|x|^{2k})|v_n|^2 = \int_{\mathbb{R}^d} \bar{v}_n(A-\lambda_n)v_n + (1+\lambda_n)\int_{\mathbb{R}^d} |v_n|^2 = o(1) \ .$$

4.2. **Radial eigenfunctions and sharpness.** This section proves remark 1.9 regarding the sharpness of corollaries 1.6 and 1.7 for some sequence (v_n) of eigenfunctions of $A = -\Delta + |x|^{2k}$ ($k \in \mathbb{N}^*$) with eigenvalues (λ_n) tending to infinity. With $v = v_n$ (n large enough) and omitting ∇ and $|x|$, (6) and (7) are the same:

$$\int_{\mathbb{R}^d} |v_n(x)|^2 dx \leqslant Ce^{C(\sqrt{\lambda_n})^{1+1/k}} \int_\Gamma |v_n(x)|^2 dx \ . \tag{26}$$

We write A in polar coordinates: $A = -\partial_r^2 - \frac{d-1}{r}\partial_r - \frac{1}{r^2}\Delta_{S^{d-1}} + r^{2k}$, where $\Delta_{S^{d-1}}$ is the Laplace-Beltrami operator on the unit sphere S^{d-1}. We denote by P_n a homogeneous polynomial of degree n which is harmonic (i.e. $\Delta P_n = 0$) and by p_n its restriction to S^{d-1}, so that p_n is a spherical harmonic with eigenvalue $\tilde{\sigma}_n = n(n+d-2)$, i.e. $-\Delta_{S^{d-1}} p_n = \tilde{\sigma}_n p_n$. We seek a sequence (v_n) of eigenfunctions of A which are radial, i.e. $v_n(r\omega) = p_n(\omega)q_n(r)$ with $r \in \mathbb{R}_+$ and $\omega \in S^{d-1}$. In order to have $v_n \in L^2(\mathbb{R}^d)$, the function $w_n(r) = r^{\frac{d-1}{2}} q_n(r)$ must be in $L^2(\mathbb{R}_+)$.

4.2.1. *Semiclassical analysis of eigenvalues.* The eigenequation bears on w_n since

$$(A-\lambda_n)v_n = (-\partial_r^2 + W_n - \lambda_n)w_n, \qquad \text{with } W_n(r) = r^{2k} + \frac{\sigma_n}{r^2},$$

where $\sigma_n = \tilde{\sigma}_n + \frac{(d-1)(d-3)}{4}$. By the change of variable $w_n(r) = \sqrt{\varepsilon_n}u_n(\varepsilon_n r)$, we shall reduce it to $(Q-\rho_n)u_n = 0$ for a semiclassical operator $Q = -h_n^2\partial_r^2 + U$ with semiclassical parameter $h_n \to 0^+$, potential U independent of n and eigenvalue ρ_n.

We omit the index n in the following computations for the sake of brevity. Since $(-\partial_r^2 w + Ww - \lambda w)(r) = \sqrt{\varepsilon}(-\varepsilon^2\partial_r^2 u + W^\varepsilon u - \lambda u)(\varepsilon r)$ with $W^\varepsilon(r) = W(\frac{r}{\varepsilon}) = \frac{r^{2k}}{\varepsilon^{2k}} + \frac{\varepsilon^2\sigma}{r^2}$, the scale ε must satisfy $\frac{1}{\varepsilon^{2k}} = \varepsilon^2\sigma$, then $h^2 = \frac{1}{\sigma}$ and $\rho = \frac{\lambda}{\varepsilon^2\sigma}$. Indeed, setting $\varepsilon = \sigma^{-\frac{1}{2(k+1)}}$, $h = \frac{1}{\sqrt{\sigma}}$ and $\rho = \lambda\sigma^{-\frac{k}{k+1}}$, the eigenequation writes

$$0 = (Q-\rho)u = -h^2\partial_r^2 u + Uu - \rho, \qquad \text{with } U(r) = r^{2k} + \frac{1}{r^2}.$$

The potential U has a unique non degenerate minimum at $r_0 = k^{-\frac{1}{2(k+1)}}$:

$$U_0 = \inf U = U(r_0) = \frac{1+\frac{1}{k}}{r_0^2} = \left(1+\frac{1}{k}\right)k^{\frac{1}{2(k+1)}} \geqslant 1.$$

From now on, we consider a sequence of eigenvalues (ρ_n) of Q which converges to U_0 (i.e. $\rho \to U_0$ as $h \to 0$) and the corresponding eigenvalues of A:

$$\lambda_n = \rho_n \sigma_n^{\frac{k}{k+1}} \underset{n}{\sim} U_0 n^{\frac{2k}{k+1}}. \tag{27}$$

Its existence results from the following harmonic approximation argument (cf. e.g. [DS99, theorem 4.23], [HN05, section 12.2]). The function $\tilde{u} \in L^2(\mathbb{R}_+)$ defined by $\tilde{u}(r_0+r) = \chi(r)h^{-\frac{1}{4}}\exp\left(-\frac{r^2E_0}{2h}\right)$, with $E_0 = \sqrt{\frac{U''(r_0)}{2}}$ and $\chi \in \mathcal{C}_0^\infty((-r_0,+\infty))$ equal to 1 near 0, is an approximate eigenfunction in the sense that it satisfies $\|(Q-U_0-hE_0)\tilde{u}\| = \|\tilde{u}\|O(h^{\frac{3}{2}})$. According to the basic spectral resolvent estimate $\|(Q-\rho)^{-1}\| \leqslant (\mathrm{dist}(\rho,\sigma(Q)))^{-1}$, this implies: $\exists\rho \in \sigma(Q)$, $\rho = U_0 + hE_0 + O(h^{\frac{3}{2}})$.

4.2.2. *Sharpness of exponents for cones.* For a cone Γ as in corollary 1.6, (26) writes

$$\int_{S^{d-1}} |p_n|^2 \leqslant Ce^{C(\sqrt{\lambda_n})^{1+1/k}} \int_{\Omega_0} |p_n|^2, \text{ since } \int_\Gamma |v_n|^2 = \int_{\Omega_0}|p_n|^2 \int_{\mathbb{R}} |w_n|^2.$$

In this paragraph we assume that there is a vector space Π of dimension 2 in $\mathbb{R}^d$ ($d \geqslant 3$) such that $\overline{\Omega}_0 \cap \Pi = \emptyset$. By a rotation of coordinates without loss of generality, $\Pi = \cap_{j=3}^d \{x_j = 0\}$ and there is an $\varepsilon \in (0,1)$ small enough such that the conic neighborhood $\Pi_\varepsilon = \left\{x \in \mathbb{R}^d \mid \sum_{j=3}^d x_j^2 \leqslant \varepsilon|x|^2\right\}$ of Π does not intersect Ω_0. Hence it is enough to consider the cone $\Gamma_\varepsilon = \left\{x \in \mathbb{R}^d \mid x_1^2 + x_2^2 \leqslant (1-\varepsilon)|x|^2\right\} \supset \Gamma$. Now we choose the harmonic polynomial $P_n(x) = r^n p_n(\omega) = (x_1+ix_2)^n$ so that $|p_n|^2 \leqslant (1-\varepsilon)^n$ on $\Omega_0^\varepsilon = S^{d-1}\cap\Gamma_\varepsilon$ (i.e. the spherical eigenfunctions (p_n) concentrate on the intersection of the plane Π with the sphere S^{d-1}), and estimate the integrals:

$$\exists\alpha_d > 0, \int_{\Omega_0^\varepsilon} |p_n|^2 \leqslant (1-\varepsilon)^n \int_{S^{d-1}} 1 = \alpha_d(1-\varepsilon)^n$$

$$\begin{aligned}
\forall\delta \in (0,1), \exists\beta_d > 0, \int_{S^{d-1}} |p_n|^2 &\geqslant \int_{B^d} (x_1^2+x_2^2)^n \\
&= \int_{B^2} (x_1^2+x_2^2)^n (1-x_1^2-x_2^2)^{\frac{d-3}{2}} \int_{B^{d-3}} 1 \\
&= \int_0^1 r^{2n}(1-r^2)^{\frac{d-3}{2}} 2\pi r dr \int_{B^{d-3}} 1 \geqslant \beta_d(1-\delta)^n.
\end{aligned}$$

Therefore, taking $\delta \in (0,\varepsilon)$ and plugging (27), (v_n) satisfies the reverse of (26):

$$\exists C>0, \forall n, \int_{\mathbb{R}^d} |v_n|^2 \geqslant \frac{\beta_d}{\alpha_d}\left(\frac{1-\delta}{1-\varepsilon}\right)^n \int_{\Gamma_\varepsilon} |v_n|^2 \geqslant Ce^{C\left(\sqrt{\lambda_n}\right)^{1+1/k}} \int_{\Gamma_\varepsilon} |v_n|^2.$$

This completes the proof that the powers in corollaries 1.6 and 1.7 are both sharp.

4.2.3. *Failure of the estimates for bounded sets.* In the last part of this section, we examine how much (26) fails when Γ is bounded. Without loss of generality, we assume that $\Gamma = B_{r_0} := \left\{x \in \mathbb{R}^d \,|\, |x| \leqslant r_0\right\}$ for some $r_0 > 0$.

Using the scaling of section 4.2.1 and setting $\delta_n = \varepsilon_n r_0$, (26) writes:

$$\int_{\mathbb{R}_+} |u_n|^2 \leqslant Ce^{\frac{C}{h_n}(\sqrt{\rho_n})^{1+1/k}} \int_0^{\delta_n} |u_n|^2, \text{ since } \int_\Gamma |v_n|^2 = \int_{S^{d-1}} |p_n|^2 \int_0^{r_0} |w_n|^2. \tag{28}$$

Since $\rho_n \to U_0$, the converse of this inequality for fixed δ_n is a typical exponential decay estimate in the classically forbidden region (cf. e.g. [EZ07, Theorem 7.3]). To disprove (28), we shall take advantage of $\delta_n \to 0$ using a semiclassical Agmon estimates (cf. e.g. [DS99, Chapter 6], [HN05, Chapter 13]).

We omit the index n in the following computations for the sake of brevity. As in [DS99, Section 6.a]), the energy equality

$$\int_{\mathbb{R}_+} |h\nabla(e^{\Phi/h}f)|^2 + \int_{\mathbb{R}_+} (U-\rho-|\Phi'|^2)|e^{\Phi/h}f|^2 = \mathrm{Re}\int_{\mathbb{R}_+} e^{2\Phi/h}Qf\overline{f},$$

holds for all $f \in \mathrm{D}(Q)$ and Φ real valued Lipschitz continuous and constant outside some compact set of $(0,+\infty)$. It implies that the second integral is nonpositive for $f = u$. In particular, if $U-\rho-|\Phi'|^2 \geqslant \frac{1}{2}$ outside some compact $K \subset (0,+\infty)$ then

$$\int_{\mathbb{R}_+\setminus K} |e^{\Phi/h}u|^2 \leqslant 2\int_K |U-\rho-|\Phi'|^2||e^{\Phi/h}u|^2. \tag{29}$$

The Agmon distance $d(r)$ from r_0 to r is the integral of $\sqrt{U-U_0}$ over the segment joining r_0 to r. Note that $d \in \mathcal{C}^1((0,+\infty))$ and $d(r) \sim -\ln r$ as $r \to 0$, since $-\sqrt{U_0} \leqslant \sqrt{U(r)-U_0} - \frac{1}{r} \leqslant 0$ for $r \leqslant \frac{1}{\sqrt{U_0}} \leqslant r_0 \leqslant 1$. As in the proof of [DS99, Theorem 6.4]), we take $\Phi(r) = \Phi_R(r) = \chi_R(\frac{1}{2}d(r))$ with $\chi_R(r) = R\chi(\frac{r}{R})$ and $\chi(r) = r$ for $r \in [0,1]$ and $\chi(r) = 1$ for $r \geqslant 1$, so that $4|\Phi_R'|^2 \leqslant |d'(r)|^2 = U - U_0$ and $U - |\Phi_R'|^2 \geqslant \frac{3}{4}(U-U_0)+U_0$. Moreover we take $K = \{r \in \mathbb{R}_+ \mid U(r) \leqslant U_0+1\}$ so that $U-\rho-|\Phi_R'|^2 \geqslant \frac{1}{2} + (\frac{1}{4}+U_0-\rho)$ on $\mathbb{R}_+ \setminus K$, and take h small enough so that $|U_0-\rho| \leqslant \frac{1}{4}$ and $]0,\delta] \cap K = \emptyset$. Using that Φ_R is nonincreasing on $[0,r_0]$ and that $|U-\rho-|\Phi_R'|^2| \leqslant \frac{5}{4}(U-U_0)+\frac{1}{4}$ and $\Phi_R(r) \leqslant \frac{1}{2}d(r)$ are bounded on K by some $C_K > 0$ independent of R, (29) implies

$$e^{2\Phi_R(\delta)/h}\int_0^\delta |u|^2 \leqslant \int_0^\delta |e^{\Phi/h}u|^2 \leqslant 2\int_K C_K|e^{C_K/h}u|^2 \leqslant 2C_Ke^{2C_K/h}\int_{\mathbb{R}_+}|u|^2.$$

As $R \to +\infty$, $2\Phi_R(\delta) \to d(\delta)$, and $-d(\delta) \sim \ln\delta \sim \ln\varepsilon \sim \frac{1}{k+1}\ln h$ as $h \to 0$, so that (28) fails by a $\ln\frac{1}{h_n}$ in the numerator, i.e. in terms of eigenfunctions of A: $\exists C_0 > 0$,

$$\int_{\mathbb{R}^d} |v_n(x)|^2 dx \geqslant C_0 e^{C_0(\sqrt{\lambda_n})^{1+1/k}\ln\lambda_n} \int_\Gamma |v_n(x)|^2 dx\ ,$$

which disproves (26) and proves that corollaries 1.6 and 1.7 fail by at least an extra logarithmic factor in the exponentials when Γ is bounded.

4.3. **Non-controllability for quadratic potential.** In this section we prove the negative part of theorem 1.10 for $k=1$. We have to disprove (9) which we rewrite:

$$\exists \kappa_T > 0, \forall v \in L^2(\mathbb{R}^d), \ \int |e^{-TA}v|^2 \leqslant \kappa_T \int_0^T \int_\Gamma |e^{-tA}v|^2 dt. \tag{30}$$

The first Hermite function $\phi_0(x) = e^{-|x|^2/2}/\pi^{d/4}$ is the normalized eigenfunction of $A = -\Delta + |x|^2$ corresponding to the smallest eigenvalue $\lambda_0 = d$. Let $T > 0$, $t_0 > 0$ and $y \in \mathbb{R}^d$. We consider the initial condition $v(x) = e^{-t_0 A}(x,y)$ so that $e^{-tA}v_0(x) = e^{-(t+t_0)A}(x,y)$ by the semigroup property. Let $T_0 = t_0 + T$. Writing the semigroup in a Hilbert basis of eigenfunctions yields

$$\int |e^{-TA}v|^2 \geqslant e^{-2T_0\lambda_0}|\phi_0(y)|^2 = C_0 e^{-|y|^2}. \tag{31}$$

4.3.1. *Non-controllability from a half-space.* In this section Γ is a half-space. Since A is the sum of d operators of the same form but in one dimension each, we may assume without loss of generality that $d=1$ and $\Gamma = (-\infty, x_0)$ for some $x_0 \in \mathbb{R}$. Mehler's explicit formula for the Hermite kernel is (cf. e.g. [Dav89, prop 4.3.1]):

$$e^{-tA}(x,y) = \frac{e^{-t}}{\sqrt{\pi(1-e^{-4t}))}} \exp\left(-\frac{(1+e^{-4t})(x^2+y^2) - 4e^{-2t}xy}{2(1-e^{-4t})}\right)$$

The function $a(t) = \frac{1+e^{-4t}}{1-e^{-4t}}$ is decreasing for $t > 0$, hence $a(T_0) > \lim_\infty a = 1$ and

$$\forall x \in \Gamma, y \in \mathbb{R}_+, t > t_0, \ |e^{-tA}(x,y)|^2 \leqslant \frac{1}{\pi(1-e^{-4t_0})} e^{-a(t_0)(x^2+y^2)+4|x_0|y/(1-e^{-4t_0})}.$$

This implies $\exists C_1 \in (1, a(T_0))$, $\exists C_2 > 0$,

$$\forall x \in \Gamma, y \in \mathbb{R}_+, t \in [t_0, T_0], |e^{-tA}(x,y)|^2 \leqslant C_2 e^{-C_2 x^2 - C_1 y^2}.$$

Therefore, setting $C_3 = TC_2 \int_{-\infty}^{x_0} e^{-C_2 x^2} dx$ yields

$$\exists C_1 > 1, \exists C_3 > 0, \forall y \in \mathbb{R}_+, \ \int_0^T \int_{-\infty}^{x_0} |e^{-tA}v|^2 dt \leqslant C_3 e^{-C_1 y^2} \tag{32}$$

The combination of (31) and (32) as $y \to +\infty$ proves that the null-controllability inequality (30) does not hold for any T.

4.3.2. *Non-controllability from any cone.* Now we do not assume that Γ is included in a half-space anymore. Indeed we consider the exterior of the revolution cone with vertex $x_0 \neq 0$, half-axis $D = \{\lambda x_0 \mid \lambda > 1\}$ and aperture angle $\theta \in (0, \pi/2)$, i.e. Γ_θ is the set of $x \in \mathbb{R}^d$ such that $x - x_0$ and x_0 make an angle greater than θ. For any $y \in D$, we define $R = |y - x_0| \sin\theta$, so that Γ_θ does not intersect the ball $B_{y,R}$ centered on y with radius R. Here we merely use that the kernel of e^{-tA} is bounded from above by the heat kernel since $A = -\Delta + |x|^2 \geqslant -\Delta$:

$$e^{-tA}(x,y) \leqslant \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

Since by definition $\Gamma_\theta \subset \mathbb{R}^d \setminus B_{y,R}$, this implies: $\exists C_1 > 0$, $\forall R > 1$,

$$\int_0^T \int_{\Gamma_\theta} |e^{-tA}v|^2 dt \leqslant \int_{t_0}^{T_0} \int_R^{+\infty} e^{-r^2/(2t)} r^{d-1} dr \frac{dt}{(4\pi t)^d} \leqslant C_1 R^{d-1} e^{-R^2/(2T_0)}.$$

Plugging this upper bound and the lower bound (31) into the null-controllability inequality (30) yields: $\forall R > 1$, $R^2/(2T_0) - (d-1)\ln R - |y|^2 \leqslant \ln(C_1/C_0)$, where $|y| = R\sin\theta + |x_0|$. Since $|y|^2 \sim |y - x_0|^2 = R^2 \sin^{-2}\theta$ as $R \to \infty$, this implies $T + t_0 = T_0 \geqslant \frac{1}{2}\sin^2\theta$. Taking $t_0 > 0$ small enough, this proves that the null-controllability inequality (30) does not hold for any $T < \frac{1}{2}\sin^2\theta$.

Appendix A. Fractional powers of semiclassical PDOs

For the sake of completeness, this appendix provides a proof of a semiclassical analysis result used in the proof of proposition 2.4 for which we do not know any exact reference. Although the chapter on functional calculus in [DS99] does not apply directly to fractional powers, we adapt it in this manner: we replace the Hellfer-Sjöstrand formula [DS99, Theorem 8.1] by the Balakrishnan-type formula (34) following [HN05, section 4.4] and [HN04, Appendix A], and we replace the estimate in [DS99, Proposition 8.6] of the symbol of the resolvent by the estimate in proposition A.2. in a symbol class depending on the spectral parameter.

We denote the semiclassical Weyl quantization of the symbol p by $\operatorname{Op}_h(p)$.

Theorem A.1. *Let $m \geqslant 1$ be an order function on $\mathbb{R}^{2d}$ and let $P = \operatorname{Op}_h(p)$ with*

(i) *$p \in S(m)$ real valued and elliptic, i.e. $p(x, \xi; h) \geqslant c_0 m(x, \xi) \geqslant c_0 > 0$,*

(ii) *P uniformly positive, i.e. $\exists c_1 > 0$, $\forall h \in (0, 1)$, $P \geqslant c_1 I$.*

Then $\forall s \in \mathbb{R}$, $\exists r \in S(m^s)$, $P^s = \operatorname{Op}_h(p^s) + h^2 \operatorname{Op}_h(r)$, hence P^s is a pseudodifferential operator with symbol in $S(m^s)$ for any semiclassical quantization rule.

The last implication results from [DS99, exercise p.83].

Under assumption (i), [DS99, pp 100–101] proves that, for all $h \in (0, h_0]$ and some sufficiently small h_0, $\operatorname{Op}_h(p^{-1})P$ is invertible, $P^{-1} \in \operatorname{Op}_h(S(\frac{1}{m}))$ and P is self-adjoint with domain $P^{-1}(L^2(\mathbb{R}^d))$. Hence the symbolic calculus and the formula $P^{-1} = \operatorname{Op}_h(p^{-1}) + P^{-1} \operatorname{Op}_h(1 - p\#_h p^{-1})$ prove the theorem in the case $s = -1$.

As in [HN05, section 4.4], the symbolic calculus now reduces the problem to s in any open interval of $\mathbb{R}$, and from now on we only consider $s \in (-1, 0)$ to take advantage of the change of variable formula

$$\forall s \in (-1, 0), \forall \lambda > 0, \quad \lambda^s = c_s \int_0^{+\infty} (\lambda + t)^{-1} t^s \, dt \,, \tag{33}$$

with $c_s = (\int_0^{+\infty} (1 + t)^{-1} t^s dt)^{-1} > 0$ (indeed $c_s = -\pi^{-1} \sin(s\pi)$ by a standard computation of one residue). Using assumption (ii) and the spectral theorem, it allows to write the fractional powers of P in terms of its resolvent:

$$\forall s \in (-1, 0), \quad P^s = c_s \int_0^{+\infty} R_t t^s dt, \quad \text{with } R_t := (P + t)^{-1} \,. \tag{34}$$

The next proposition describes a symbol class relevant to R_t. We say that a nonnegative function m_t on $\mathbb{R}^{2d}$ with parameter t is an order function uniformly with respect to t when $\exists C_0$, $N_0 > 0$, $\forall X$, $Y \in \mathbb{R}^{2d}$, $\forall t$, $m_t(X) \leqslant C_0 \langle X - Y \rangle^{N_0} m_t(Y)$, where $\langle X \rangle = \sqrt{1 + |X|^2}$. We say that the symbol $p_t \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ with parameter t is in the class $S(m_t)$ uniformly with respect to t when $\forall \alpha \in \mathbb{N}^{2d}$, $\exists C_\alpha > 0$, $\forall X \in \mathbb{R}^{2d}$, $\forall t$, $|\partial^\alpha p_t(X)| \leqslant C_\alpha m_t(X)$.

Proposition A.2. *Under the assumptions of theorem A.1, the symbolic product $q_t = (p+t)\#_h(p+t)^{-1}$ satisfies $h^{-2}(1 - q_t) \in S(1)$ uniformly with respect to $t \geqslant 0$. Moreover there exists $r_t \in S((m + t)^{-1})$ uniformly with respect to $t \geqslant 0$ such that $R_t = \operatorname{Op}_h(r_t)$ for all $t \geqslant 0$ and $h \in (0, h_0]$, for some $h_0 > 0$ independent of t.*

Proof. Uniformly with respect to $t \geqslant 0$, $m + t$ is an oder function, $p + t \in S(m + t)$ and $(p + t)^{-1} \in S((m + t)^{-1})$. By the continuity of the symbolic product in [DS99, Proposition 7.7], we deduce $q_t \in S(1)$ uniformly with respect to $t \geqslant 0$. But the first two terms of the asymptotic expansion of q_t are $(p + t)(p + t)^{-1} = 1$ and $\frac{h}{2} \left\{p + t, (p + t)^{-1}\right\} = 0$. Hence $h^{-2}(1 - q_t) \in S(1)$ uniformly with respect to $t \geqslant 0$.

We deduce that $\operatorname{Op}_h(q_t) = I - h^2 \operatorname{Op}_h(h^{-2}(1 - q_t))$ is invertible as an L^2 bounded operator for all $t \geqslant 0$ and $h \in (0, h_0)$, for some $h_0 > 0$ independent of t. As in the proof of [DS99, (8.10)], the semiclassical Beal's characterization in [DS99, Proposition 8.3] (which is still valid with a parameter t as long as all the estimates are uniform with respect to t) implies that $\mathrm{Op}_h(q_t)^{-1} = \mathrm{Op}_h(\tilde{q}_t)$ with $\tilde{q}_t \in S(1)$ uniformly with respect to $t \geqslant 0$. Since the definition of q_t as symbolic product also writes $\mathrm{Op}_h(q_t) = R_t^{-1}\,\mathrm{Op}_h((p+t)^{-1})$, we deduce $R_t = \mathrm{Op}_h((p+t)^{-1})\,\mathrm{Op}_h(q_t)^{-1} = \mathrm{Op}_h(r_t)$ with $r_t = (p+t)^{-1}\#_h\tilde{q}_t \in S((m+t)^{-1})$ uniformly with respect to $t \geqslant 0$. □

Combining (34) and (33) with $\lambda = p$ yields

$$\forall s \in (-1,0), \quad P^s - \mathrm{Op}_h(p^s) = c_s \int_0^{+\infty} \mathrm{Op}_h(r_t)\,\mathrm{Op}_h(1-q_t) t^s dt \; .$$

By proposition A.2, we deduce $P^s - \mathrm{Op}_h(p^s) = h^2\,\mathrm{Op}_h(r)$ with $r = c_s \int_0^{+\infty} a_t t^s dt$ and $a_t = h^{-2} r_t \#_h (1-q_t) \in S((m+t)^{-1})$ uniformly with respect to $t \geqslant 0$. But $\int_0^{+\infty} |\partial^\alpha a_t| t^s dt < \infty$ since $(m+t)^{-1} \leqslant (1+t)^{-1}$ and $s \in (-1,0)$. Hence differentiating r under the integral sign and (33) with $\lambda = m$ prove $r \in S(m^s)$, which completes the proof of theorem A.1 for $s \in (-1,0)$.

References

[BHLR09] F. Boyer, F. Hubert, and J. Le Rousseau, *Discrete Carleman estimates for elliptic operators and uniform controllability of semi-discretized parabolic equations*, http://hal.archives-ouvertes.fr/hal-00366496/, 2009.

[Bur98] N. Burq, *Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel*, Acta Math. **180** (1998), no. 1, 1–29.

[Bur02a] ———, *Lower bounds for shape resonances widths of long range Schrödinger operators*, Amer. J. Math. **124** (2002), no. 4, 677–735.

[Bur02b] ———, *Semi-classical estimates for the resolvent in nontrapping geometries*, Int. Math. Res. Not. (2002), no. 5, 221–241.

[Bur04] ———, *Smoothing effect for Schrödinger boundary value problems*, Duke Math. J. **123** (2004), no. 2, 403–427.

[BZ04] N. Burq and M. Zworski, *Control in the presence of a black box*, J. Amer. Math. Soc. **17** (2004), 443–471.

[CDMZ01] V. R. Cabanillas, S. B. De Menezes, and E. Zuazua, *Null controllability in unbounded domains for the semilinear heat equation with nonlinearities involving gradient terms*, J. Optim. Theory Appl. **110** (2001), no. 2, 245–264.

[Chr07] H. Christianson, *Semiclassical non-concentration near hyperbolic orbits*, J. Funct. Anal. **246** (2007), no. 2, 145–195.

[Dav89] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989.

[DM09] T. Duyckaerts and L. Miller, *Resolvent conditions for the controllability of parabolic semigroups*, manuscript, 2009.

[DS84] E. B. Davies and B. Simon, *Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians*, J. Funct. Anal. **59** (1984), no. 2, 335–395.

[DS99] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Mathematical Society Lecture Note Series, vol. 268, Cambridge Univ. Press, 1999.

[Èma95] O. Yu. Èmanuilov, *Controllability of parabolic equations*, Mat. Sb. **186** (1995), no. 6, 109–132.

[EZ07] L. C. Evans and M. Zworski, *Lectures on semiclassical analysis*, Version 0.3. http://math.berkeley.edu/~zworski/semiclassical.pdf, 2007.

[FCZ00] E. Fernández-Cara and E. Zuazua, *The cost of approximate controllability for heat equations: the linear case*, Adv. Differential Equations **5** (2000), no. 4-6, 465–514.

[GBdT07] M. González-Burgos and L. de Teresa, *Some results on controllability for linear and nonlinear heat equations in unbounded domains*, Adv. Differential Equations **12** (2007), no. 11, 1201–1240.

[HN04] F. Hérau and F. Nier, *Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential*, Arch. Ration. Mech. Anal. **171** (2004), no. 2, 151–218.

[HN05] B. Helffer and F. Nier, *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*, Lecture Notes in Mathematics, vol. 1862, Springer-Verlag, Berlin, 2005.

[Hör63] L. Hörmander, *Linear partial differential operators*, Die Grundlehren der mathematischen Wissenschaften, Bd. 116, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[JL99] D. Jerison and G. Lebeau, *Nodal sets of sums of eigenfunctions*, Harmonic analysis and partial differential equations (Chicago, 1996), Univ. Chicago Press, 1999, pp. 223–239.

[KT05] H. Koch and D. Tataru, *L^p eigenfunction bounds for the Hermite operator*, Duke Math. J. **128** (2005), no. 2, 369–392.

[KTZ07] H. Koch, D. Tataru, and M. Zworski, *Semiclassical L^p estimates*, Ann. Henri Poincaré **8** (2007), no. 5, 885–916.

[LR95] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur*, Comm. Partial Differential Equations **20** (1995), no. 1-2, 335–356.

[LR07] J. Le Rousseau, *Représentation microlocale de solutions de systèmes hyperboliques, application à l'imagerie, et contributions au contrôle et aux problèmes inverses pour des equations paraboliques*, HDR. Université de Provence - Aix-Marseille I. http://tel.archives-ouvertes.fr/tel-00201887/, 2007.

[LRL09] J. Le Rousseau and G. Lebeau, *Introduction aux inégalités de Carleman pour les opérateurs elliptiques et paraboliques. Applications au prolongement unique et au contrôle des équations paraboliques*, http://hal.archives-ouvertes.fr/hal-00351736/, 2009.

[LRR] J. Le Rousseau and L. Robbiano, *Carleman estimate for elliptic operators with coefficients with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations*, to appear in Arch. Rational Mech. Anal., http://hal.archives-ouvertes.fr/hal-00193885.

[LZ98] G. Lebeau and E. Zuazua, *Null-controllability of a system of linear thermoelasticity*, Arch. Rational Mech. Anal. **141** (1998), no. 4, 297–329.

[Mar06] J. Marzuola, *Eigenfunctions for partially rectangular billiards*, Comm. Partial Differential Equations **31** (2006), no. 4-6, 775–790.

[Mil05a] L. Miller, *Controllability cost of conservative systems: resolvent condition and transmutation*, J. Funct. Anal. **218** (2005), no. 2, 425–444.

[Mil05b] ______, *On the controllability of the heat equation in unbounded domains*, Bull. Sci. Math. **129** (2005), no. 2, 175–185.

[Mil05c] ______, *Unique continuation estimates for the Laplacian and the heat equation on non-compact manifolds*, Math. Res. Lett. **12** (2005), no. 1, 37–47.

[Mil06] ______, *On the controllability of anomalous diffusions generated by the fractional Laplacian*, Math. Control Signals Systems **18** (2006), no. 3, 260–271.

[Mil09] ______, *A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups*, http://hal.archives-ouvertes.fr/hal-00411846/, 2009.

[MS07] G. Metafune and C. Spina, *Kernel estimates for a class of Schrödinger semigroups*, J. Evol. Equ. **7** (2007), no. 4, 719–742.

[MZ01] S. Micu and E. Zuazua, *On the lack of null-controllability of the heat equation on the half-line*, Trans. Amer. Math. Soc. **353** (2001), no. 4, 1635–1659.

[Van09] M. VanValkenburgh, *Exponential lower bounds for quasimodes of semiclassical Schrödinger operators*, Math. Res. Lett. **16** (2009), no. 4, 721–734.

[VZ08] J. Vancostenoble and E. Zuazua, *Null controllability for the heat equation with singular inverse-square potentials*, J. Funct. Anal. **254** (2008), no. 7, 1864–1902.

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