On the cost of fast control for heat-like semigroups: spectral inequalities and transmutation

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Abstract

The Lebeau-Robbiano strategy deduces the null-controllability of parabolic systems of linear PDEs from some spectral inequalities: we implement it so simply that it yields valuable estimates on the cost of the distributed control as the time available to perform it tends to zero. The solutions and controls of some parabolic equations can be expressed in terms of those of the hyperbolic equation obtained by changing the time derivative into a second oder derivative: such transmutation formulas yield geometric bounds on both the spectral and cost rates.

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1 Heat-like control problems

The basic null-controllability problem we shall consider is: for any given initial temperature distribution in a domain M, find a heat source located in a given region Ω such that the temperature is null everywhere after heat has diffused during a given time T. The cost is the ratio κ_T of the size of the input over the size of the initial state: it blows up as T tend to zero.

In § 2, we present the quite general method introduced in [Mil09] to simultaneously prove null-controllability and estimate this blow-up. For this basic problem, it proves that the cost κ_T blows up like $\exp(c/T)$ and it is based on an observability estimate for sums of eigenfunctions of the Dirichlet Laplacian proved by some Carleman estimates in joint papers of Lebeau with Jerison, Robbiano and Zuazua. Moreover, it bounds the rate c from above in terms of the rate a in this estimate.

In § 3, we present some bounds on the spectral rate a and the cost rate c in terms of distances related to the configuration of the control region Ω within the domain M. The lower bounds are based on Gaussian estimates in L^2 rather than pointwise. This approach introduced in [Mil09] also simplifies and generalizes earlier results. The upper bound on c is based on the control transmutation method, cf. [Mil06].

Here are some control problems to which these results apply, cf. [Mil09].

Notations: *We denote in the same way an open subset $\Omega \neq \emptyset$ of M, its characteristic function and the multiplication by this function which is a bounded operator on the Hilbert space $\mathcal{H} = L^2(M)$ with norm $\|\cdot\|$.* For simplicity, we always consider Dirichlet condition on the boundary ∂M of M, we solve equations in Hilbert spaces of square integrable functions and achieve control by input functions u in $L^2([0,T]\times M) = L^2([0,T];\mathcal{H})$.

Problem 1: Consider the heat equation as an ODE in $\mathcal{H}$,

$$\partial_t f - \Delta f = \Omega u,$$

where $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2$ is the usual Laplacian and M is a smooth connected bounded domain in $\mathbb{R}^d$. (Δ could also be the Laplace-Beltrami operator on a compact smooth connected Riemannian manifold M.)

Problem 2: Consider the same heat equation where $M = \mathbb{R}^d$ and Ω is the exterior of a ball with center 0 and radius R. (Δ could also be the Laplace-Beltrami operator on a complete smooth connected Riemannian manifold M with Ω the exterior of a compact K such that $K \cap \overline{\Omega} \cap \partial M = \emptyset$.)

Problem 3: Consider the diffusion equation,

$$\partial_t f - Pf = \Omega u, \quad Pf = \sum_{j,k=1}^{d} \partial_{x_j}(g_{jk}\partial_{x_k} f) + \sum_{j=1}^{d} \partial_{x_j}(b_j f) + Vf,$$

where M is a C^2 connected bounded domain in $\mathbb{R}^d$, b_j and V are complex valued and bounded on M, $g_{jk} \in C^1(M)$, the matrix $G = (g_{ij})$ is real symmetric and $0 < G \leqslant I$ uniformly on M.

Problem 4: Consider the linear Ginzburg-Landau equation,

$$(1+\rho i)\partial_t f + (-\Delta + V) f = \Omega u,$$

where $V \in L^1_{loc}(M)$ is a real potential bounded from below, $\rho \in \mathbb{R}$, and M as in problems 1 or 2.

Problem 5: Consider thermoelastic plates without rotatory inertia,

$$\begin{cases} \partial_t^2 g + \Delta^2 g + \alpha \Delta f = \Omega u_1 & \text{on } (0,T)\times M, \\ \partial_t f - \Delta f - \alpha \Delta \partial_t g = \Omega u_2 & \text{on } (0,T)\times M, \\ g = \Delta g = f = 0 & \text{on } (0,T)\times \partial M, \end{cases}$$

for § 2, where either $u_1 = 0$ or $u_2 = 0$, $\alpha > 0$, M and Ω are as in problem 1.

2 The direct Lebeau-Robbiano strategy

Each of these problems can be formulated in terms of a continuous semigroup $(e^{tA})_{t\geqslant 0}$ on a Hilbert space with suitable generator A. By a duality argument, null-controllability is equivalent to the *final-observability inequality* (stated here for problems 1 to 4, but similar in problem 5),

$$\|e^{TA}\phi\|^2 \leqslant \kappa_T \int_0^T \|\Omega e^{tA}\phi\|^2 dt, \quad \phi \in \mathcal{H}, \quad T > 0, \tag{FinalObs}$$

where κ_T is the cost, i.e. the ratio of the size $\int_0^T \|u\|^2 dt$ of the input over the size $\|f(0)\|^2$ of the initial state which it annihilates.

In problem 1, $A = \Delta$ with domain $D(A) = H^2(M) \cap H_0^1(M) \subset \mathcal{H}$ and the Lebeau-Robbiano strategy is based on the *spectral inequality*,

$$\|v\| \leqslant a_0 e^{a\lambda} \|\Omega v\|, \quad \lambda > 0, \quad v \in \mathcal{H}_\lambda, \tag{SpecObs}$$

where $\mathcal{H}_\lambda$ denotes the set of finite sums of eigenfunctions of $-\Delta$ with eigenvalues less than λ^2. More generally $\mathcal{H}_\lambda$ is the spectral subspace of $\mathcal{H}$ relative to the part of the spectrum of A with real part greater than $-\lambda^2$. E.g. (SpecObs) holds for problem 2: $v \in \mathcal{H}_\lambda$ means that the Fourier transform $\hat{v}(\xi)$ vanishes for $|\xi| > \lambda$, i.e. v is the restriction to the real axis of an entire function $\tilde{v}$ such that $|\tilde{v}(z)| \leqslant c e^{\lambda |\operatorname{Im} z|}$ by the Paley-Wiener theorem.

Theorem 1 (SpecObs) *implies* (FinalObs) *with, for any* $c > 4a^2$*, the cost bound:* $\kappa_T \leqslant c_0 e^{2c/T}$.

With $c > 8a^2$, this theorem is due to Seidman. Indeed [Sei08] states an abstract theorem which also applies to problem 5. The statement in [Mil09] has a simpler cost bound and is more general (e.g. it applies to [Lea09] and to linear elastic systems with structural damping for which λ in (SpecObs) and T in the cost bound have fractional powers). In his survey [Zua06], which covers the Lebeau-Robbiano strategy and the cost of fast controls, Zuazua remarked: “Actually, as far as we know, there is no direct proof of the fact that the spectral observability inequality implies the observability inequality for the heat equation. The existing proof is that due to Lebeau and Robbiano and passes through the property of null controllability and duality”. Whereas Seidman’s proof of this theorem still uses approximate null-controllability, our *direct* proof does not have recourse to controllability. (cf. [TT09] for a less general direct proof based on the moment problem)

For problem 3, with V real and $b_j = 0$ to ensure self-adjointness, (SpecObs) is proved in [BHLR09] and theorem 1 still holds, including with ∂_t replaced by $(1 + \rho i)\partial_t$ as in problem 4. The cost bound in theorem 1 is still valid for problem 5, keeping (SpecObs) as in problem 1 and replacing Ω and $\mathcal{H}$ in (FinalObs) by the suitable bounded observation operator.

We explain the direct Lebeau-Robbiano strategy on a simple example.

Proof that (SpecObs) with $a=1$ implies the cost bound $\kappa_T \leqslant e^{18/T}$, for T small enough, for problems 1 and 2. First note that $\kappa_T \leqslant 1/T \leqslant e^{1/T}$ when $\Omega = M$. Hence (SpecObs) with $a=1$ implies

$$\|e^{T\Delta}v\|^2 \leqslant e^{\lambda+1/T}\int_0^T \|\Omega e^{t\Delta}v\|^2 dt, \quad T>0, \quad \lambda>0, \quad v\in\mathcal{H}_\lambda.$$

On the other hand, for $w \perp \mathcal{H}_\lambda$ the time-decay rate increases with λ: $\|e^{t\Delta}w\| \leqslant e^{-t\lambda^2}\|w\|$. To prove (FinalObs), we decompose $\phi = v + w$ with $\lambda = 3/T$, and use $\|e^{T\Delta}\phi\|^2 = \|e^{T\Delta}v\|^2 + \|e^{T\Delta}w\|^2 \leqslant \|e^{T\Delta}v\|^2 + e^{-9/T}\|w\|^2$. Since there is no decay as $t\to 0$, we observe on $[T/2, T]$ but not on $[0, T/2]$:

$$\begin{cases} \|e^{T\Delta}v\|^2 \leqslant e^{\lambda+2/T}\int_{\frac{T}{2}}^T \|\Omega e^{t\Delta}v\|^2 dt = e^{5/T}\int_{\frac{T}{2}}^T \|\Omega e^{t\Delta}v\|^2 dt, & v\in\mathcal{H}_\lambda, \\ \int_{\frac{T}{2}}^T \|\Omega e^{t\Delta}w\|^2 dt \leqslant \frac{T}{2}\|e^{\frac{T}{2}\Delta}w\|^2 \leqslant Te^{-T\lambda^2}\|w\|^2 = Te^{-9/T}\|w\|^2, & w\perp\mathcal{H}_\lambda. \end{cases}$$

Hence $\|e^{T\Delta}\phi\|^2 \leqslant 2e^{5/T}\left(\int_{\frac{T}{2}}^T \|\Omega e^{t\Delta}\phi\|^2 dt + Te^{-9/T}\|w\|^2\right) + e^{-9/T}\|w\|^2$.

Taking T small enough, setting $\tau = T$, $q = 2/3$ and $f(T) = e^{-6/T}$, this yields the approximate observability inequality,

$$f(\tau)\|e^{\tau\Delta}\phi\|^2 - f(q\tau)\|\phi\|^2 \leqslant \int_0^\tau \|\Omega e^{t\Delta}\phi\|^2 dt, \quad \phi\in\mathcal{H}, \quad \tau\in(0,\varepsilon_0). \quad (1)$$

Consider the partition $(0,T] = \cup_{k\in\mathbb{N}}(T_{k+1}, T_k]$ with $T_{k+1} - T_k = \tau_k = q\tau_{k-1}$. Applying (1) on each $[T_{k+1}, T_k]$ with $\tau = \tau_k$ and adding these telescoping inequalities yield, since $f(T)\to 0$ as $T\to 0$:

$$f((1-q)T)\|e^{T\Delta}\phi\|^2 = f(\tau_0)\|e^{T_0\Delta}\phi\|^2 - 0\times\|\phi\|^2 \leqslant \int_0^T \|\Omega e^{t\Delta}\phi\|^2 dt.$$

This completes the proof of (FinalObs) with $\kappa_T \leqslant 1/f((1-q)T) = e^{18/T}$.

3 Transmutations

The lower bounds in this section are given in terms of the following distance:

$$d_\Omega = \sup_{y\in M} \operatorname{dist}(\overline{\Omega}, y),$$

i.e. the furthest from Ω a point of M can be, e.g. in problem 2: $d_\Omega = R$.

Theorem 2 (FinalObs) *with* $\kappa_T = c_0 e^{2c/T}$ *implies* $c \geqslant d_\Omega^2/4$.

Theorem 3 (SpecObs) *implies* $a \geqslant d_\Omega/2$.

Theorem 2 holds for problems 1 to 4. Theorem 3 holds for problems 1 to 4 with $b_j = 0$ and V real to ensure selfadjointness and natural spaces $\mathcal{H}_\lambda$.

The upper bound in theorem 4 is given in terms of the length L_Ω of the longest generalized geodesic in M which does not intersect Ω (these geodesics behave like the rays of geometrical optics), e.g. in problem 2: $L_\Omega = 2R$.

When M is a segment of length L it is known that null-controllability with input at one end holds with the cost bound $\kappa_T \leqslant c_0 e^{2c/T}$ for any $c > c_* L^2$. Tenenbaum and Tucsnak proved in 2007 that $c_* \leqslant 3/4$.

Theorem 4 (FinalObs) *holds with* $\kappa_T \leqslant c_0 e^{2c/T}$ *for any* $c > c_* L_\Omega^2$.

N.b. theorem 4 is only relevant when L_Ω is finite. The proof of theorem 4 is based on this result of Bardos, Lebeau and Rauch: $L > L_\Omega$ is sufficient (and almost necessary) for the exact controllability of the wave equation $\partial_t^2 g - \Delta g = \Omega u$ in the context of problem 1. This result, hence theorem 4, extend to problem 2. Burq extended this result to $\partial_t^2 g - Pg = \Omega u$, when M is C^3, P is as in problem 3 with $g_{jk} \in C^2(M)$, $b_j = 0$ and $V = 0$, replacing the geodesics by bicharacteristics of G, hence theorem 4 holds under these assumptions.

The key idea in this section is that the wave equation $\partial_t^2 w = Aw$ describes "the geometry" of the diffusion equation $\partial_t \phi = A\phi$ in short times. For $A = \Delta$, Kannai implemented this idea with the *transmutation formula* (in order to prove an asymptotic expansion of the heat kernel as $t \to 0$),

$$e^{t\Delta}\phi = \int_{-\infty}^{+\infty} k(t,s)\cos(s\sqrt{-\Delta})\phi ds, \quad k(t,s) = \frac{1}{\sqrt{4\pi t}} e^{-s^2/(4t)}, \tag{2}$$

which gives the solution of the heat equation as a weighted integral over all times of the (even) solution of the wave equation with the same initial condition. This formula results directly from the integral representation of functions of Δ via spectral measures and the Fourier transform of the Gaussian. Cheeger, Gromov and Taylor fully exploited such transmutations in combination with the finite speed of propagation.

Indeed the support of these waves propagate with speed less than one, i.e. $\Omega\cos(t\sqrt{-\Delta})\omega = 0$, $t \in (0,\delta)$, where ω and Ω are open subsets of M and $\delta = \operatorname{dist}(\Omega,\omega)$. Plugging this in (2) implies the L^2 *Gaussian estimate* on which theorems 1 and 2 are based,

$$\|\Omega e^{t\Delta}\omega\| \leqslant \|\omega\| \frac{1}{\sqrt{4\pi t}} \int_{|s|\geqslant\delta} e^{-s^2/(4t)} ds \leqslant \|\omega\| e^{-\delta^2/(4t)}. \tag{3}$$

Proof of theorem 2 for problems 1 and 2: Given $\delta < d_\Omega$, there is an open ball $\omega \subset M$ such that $\operatorname{dist}(\Omega,\omega) = \delta$. Taking $\phi = \omega$ in (SpecObs), applying (3) and taking the limit $T \to 0$ yields a contradiction for $c \geqslant \delta^2/4$:

$$0 \neq \|\omega\|^2 \leftarrow \|e^{TA}\phi\|^2 \leqslant c_0 e^{\frac{2c}{T}} \int_0^T \|\Omega e^{tA}\phi\|^2 dt \leqslant T c_0 d_0^2 \|\omega\|^2 e^{\frac{2(c-\delta^2/4)}{T}} \to 0.$$

Hence $c > \delta^2/4$. Taking the limit $\delta \to d_\Omega$ completes the proof of theorem 2.

The proof of theorem 3 is similar (Jerison and Lebeau proved $a > 0$ for problem 1 by a pointwise Gaussian estimate and Weyl's law for eigenvalues).

The proof of theorem 4 uses the *control transmutation method*, cf. [Mil06]. It replaces the one dimensional fundamental heat solution k in (2) by the solution of the heat equation obtained from an initial Dirac mass with an input on the boundary of $[-L, L]$ which performs null-controllability in time T. Thus both the solution and the input of the wave equation are transmuted into those of the heat equation (this idea was first used by Phung for Schrödinger equation). Cf. [MZ09] for a recent numerical application.

An upper bound for a in problem 1 with $d = 1$ is given in [TT09].

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