Fast control cost for heat-like semigroups: Lebeau-Robbiano strategy and Hautus test

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Abstracts

Fast control cost for heat-like semigroups: Lebeau-Robbiano strategy and Hautus test

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(joint work with Thomas Duykaerts)

Since the seminal work of Russell and Weiss in [7], resolvent conditions for various notions of admissibility, observability and controllability, and for various notions of linear evolution equations have been studied intensively, sometimes under the name of infinite-dimensional Hautus test, cf. [8, 2]. This talk based on [1] investigates resolvent conditions for null-controllability in arbitrary time: necessary conditions for general semigroups, and sufficient conditions for analytic normal semigroups and semigroups with negative self-adjoint generators.

1. Introduction

Let $-A$ be the generator of a strongly continuous semigroup on a Hilbert space $\mathcal{E}$. Let C be a bounded operator from the domain $\mathrm{D}(A)$ with the graph norm to another Hilbert space $\mathcal{F}$. The norms in $\mathcal{E}$ and $\mathcal{F}$ are denoted $\|\cdot\|$. We refer to the monograph [8] for a full account of the control theory of semigroups.

Recall the usual *admissibility* condition (for some time $T > 0$ hence all $T > 0$),

$$\exists K_T > 0,\ \forall v \in \mathrm{D}(A), \quad \int_0^T \|Ce^{-tA}v\|^2 dt \leq K_T \|v\|^2. \tag{1}$$

If C is admissible for A then null-controllability at time T is equivalent to *final-observability* at time T (cf. [8], i.e.

$$\exists \kappa_T > 0,\ \forall v \in \mathcal{E}, \quad \|e^{-TA}v\|^2 \leq \kappa_T \int_0^T \|Ce^{-tA}v\|^2 dt. \tag{2}$$

The control property investigated here is (2) *for all* $T > 0$.

1.1. **Control cost.** The coefficient κ_T in (2) is the *control cost*: it is the ratio of the size of the input over the size of the initial state which the input steers to the zero final state in a lapse of time T. It blows up as $T \to 0$. E.g. for the heat semigroup on a compact manifold M with Dirichlet boundary conditions observed from a subset Ω: $\kappa_T \leq c_0 \exp(2c/T)$, $T \in (0, 1)$, where c_0 is a positive constant, implies $c \geq d^2/4$ where d is the furthest a point of M can be from Ω, and is implied by $c > 3L^2/4$ where L is the length of the longest generalized geodesic in M which does not intersect Ω ($L < +\infty$ is known as the condition of Bardos-Lebeau-Rauch).

For many evolutions of parabolic type, κ_T is bounded by $c_0 \exp(2c/T^\beta)$ where c, c_0 and β are positive constants. E.g. thermoelastic plates without rotatory inertia, the plate equation with square root damping, diffusions in discontinuous media or in a potential well, diffusions generated by the fractional Laplacian or non-selfadjoint elliptic generators, cf. references in [5].

1.2. **Resolvent conditions.** The resolvent condition: $\exists M > 0$,

$$\|v\|^2 \leq \frac{M}{(\operatorname{Re}\lambda)^2}\|(A-\lambda)v\|^2 + \frac{M}{\operatorname{Re}\lambda}\|Cv\|^2, \quad v \in \mathrm{D}(A), \quad \operatorname{Re}\lambda > 0. \tag{3}$$

was introduced in [7] as a necessary condition for exact observability in infinite time of exponentially stable semigroups.

When A is skew-adjoint (equivalently when the semigroup is a unitary group), it was proved in [3] that the following resolvent condition is necessary and sufficient for final-observability (hence exact observability) in some time $T > 0$: $\exists M > 0$,

$$\|v\|^2 \leq M\|(iA-\lambda)v\|^2 + M\|Cv\|^2, \quad v \in \mathrm{D}(A), \quad \lambda \in \mathbb{R}.$$

We refer to [6] for more background and references. This result was extended to some more general groups in [2, theorem 1.2].

When $-A$ generates an exponentially stable normal semigroup, [2, theorem 1.3] proves that the resolvent condition (3) is sufficient for the weaker notion:

$$\exists T > 0, \exists \kappa_T > 0, \forall v \in \mathcal{E}, \|e^{-TA}v\|^2 \leq \kappa_T \int_0^\infty \|Ce^{-tA}v\|^2 dt. \tag{4}$$

In this framework (4) implies (2) for *some* time T.

But it seems that resolvent conditions for final-observability for *any* $T > 0$ in (2) has not been investigated yet, although it is quite natural for heat-like semigroups.

2. Results

2.1. **Necessary resolvent conditions for semigroups.** The proof mainly consists in changing i into -1 in [3, lemma 5.2]. Cf. also the proof of [7, theorem 1.2].

Theorem 1. *Let $B_T = \sup_{t\in[0,T]} \|e^{-tA}\|$ be the semigroup bound up to time T. If (1) and (2) hold then : $\forall v \in \mathrm{D}(A)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$,*

$$\|v\|^2 \leq 2e^{2T\operatorname{Re}\lambda}\left((B_T^2 + 2\kappa_T K_T)\frac{\|(A-\lambda)v\|^2}{(\operatorname{Re}\lambda)^2} + \kappa_T\frac{\|Cv\|^2}{\operatorname{Re}\lambda}\right),$$

Theorem 2. *If final-observability (2) holds for all $T \in (0, T_0]$ with the control cost $\kappa_T = c_0 e^{\frac{2c}{T^\beta}}$ for some positive β, c and c_0 then the resolvent condition*

$$\|v\|^2 \leq a_0 e^{2a(\operatorname{Re}\lambda)^\alpha}\left(\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathrm{D}(A), \quad \operatorname{Re}\lambda > 0,$$

holds with power $\alpha = \frac{\beta}{\beta+1}$ and rate $a = c^{\frac{1}{\beta+1}}\frac{\beta+1}{\beta^\alpha}$.

It still holds for $\lambda \in \mathbb{C}$ if $\operatorname{Re}\lambda$ is replaced by $\operatorname{Re}_+\lambda := \max\{\operatorname{Re}\lambda, 0\}$.

2.2. **Sufficient resolvent conditions for an analytic normal semigroup.** The proof is based on the Lebeau-Robbiano strategy of [5]. N.b. (1) is not assumed.

Theorem 3. *Assume that $-A$ generates an analytic normal semigroup, hence there exists $\omega \in \mathbb{R}$ and $\theta \in [0, \frac{\pi}{2})$ such that $\sigma(A) \subset \{z \in \mathbb{C} : \arg(z-\omega) \leq \theta\}$.*

The resolvent condition with $\alpha \in (0,1)$, $\omega_0 < \omega$, $\lambda_0 > \omega_0$, positive a_0 and a,

$$\|v\|^2 \leq \frac{\cos^2\theta}{(\lambda-\omega_0)^2}\|(A-\lambda)v\|^2 + a_0 e^{2a\lambda^\alpha}\|Cv\|^2, \quad v \in \mathrm{D}(A), \quad \lambda \geq \lambda_0,$$

implies final-observability (2) *for all time* $T > 0$ *with the control cost estimate*

$$\limsup_{T\to 0} T^\beta \ln \kappa_T \leq 2a^{\beta+1}(\beta+1)^{\beta(\beta+1)}\beta^{-\beta^2}, \qquad \text{where } \beta = \frac{\alpha}{1-\alpha}.$$

2.3. **Sufficient resolvent condition for a negative self-adjoint generator.** The proof combines the Lebeau-Robbiano strategy of [5], the control transmutation method of [4] (which deduces the final-observability of the heat-like equation $\dot{v} + Av = 0$ from the exact observability of the wave-like equation $\ddot{w} + Aw = 0$) and results on resolvent conditions from [6].

Theorem 4. *Assume that the positive self-adjoint operator* A *and the operator* C *bounded from* $\mathrm{D}(\sqrt{A})$ *with the graph norm to* $\mathcal{F}$ *satisfy the admissibility and observability conditions with nonegative powers* γ *and positive* δ, L_* *and* M_**:*

$$\|Cv\|^2 \leq L_* \lambda^\gamma \left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|v\|^2\right), \quad v \in \mathrm{D}(A), \quad \lambda \geq \inf A,$$

$$\|v\|^2 \leq M_* \lambda^\delta \left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathrm{D}(A), \quad \lambda \geq \inf A.$$

If $\gamma + \delta < 1$ *then final-observability* (2) *holds for all* $T > 0$ *with the cost estimate*

$$\limsup_{T\to 0} T^\beta \ln \kappa_T < +\infty, \qquad \text{where } \beta = \frac{1+\gamma+\delta}{1-\gamma-\delta}.$$

The assumption of the control transmutation method corresponds to $\gamma = \delta = 0$. The Russell-Weiss condition (3) corresponds to $\delta = -1$.

A logarithmic improvement of this theorem is also proved in [1] thanks to this new variant of the Lebeau-Robbiano strategy of [5]:

Theorem 5. *Assume the admissibility condition* (1) *and that* $-A$ *generates a normal semigroup. If the logarithmic observability condition on spectral subspaces*

$$\|v\|^2 \leq a_0 e^{2a\lambda/((\log(\log\lambda))^\alpha \log\lambda)} \|Cv\|^2, \quad v \in \mathbf{1}_{\mathrm{Re}\, A<\lambda}\mathcal{E}, \quad \lambda \geq \lambda_0.$$

holds with $\alpha > 2$, λ_0, a_0, *a positive then final-observability* (2) *holds for all* $T > 0$.

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